Chern-Weil Theory and Characteristic Classes

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Contents

1 Geometry of Hermitian Vector Bundles

1.1 Hermitian vector bundles and metrics

Let $\pi : E \to X$ be a complex rank k bundle over some real manifold X. We do not assume for the moment that X has an almost complex structure. Let $\Gamma(U, E)$ denotes the vector space of all smooth sections of E over U.

In this section, vector bundles are all refered to differentiable complex vector bundles over a differentiable manifold, $E \to X$.

Definition 1.1. *Let $E \to X$ be an complex vector bundle of rank r and let U be an open subset of X. A (**moving**) **frame** for E over U is a set of r smooth sections $\{s_1, \cdots, s_r\}$, $s_j \in \Gamma(U, E)$, such that $\{s_1(x), \cdots, s_r(x)\}$ is a basis for E_x for any $x \in U$.*

Proposition 1.2. *Any complex vector bundle E admits a frame in some neighborhood of any given point in the base space.*

Proof. Let U be a trivializing neighborhood for E so that

$$h : E|_U \xrightarrow{\sim} U \times \mathbb{C}^r$$

is a bundle chart. Thus we have an isomorphism

$$h_* : \Gamma(U, E|_U) \to \Gamma(U, U \times \mathbb{C}^r).$$

Consider the vector-valued functions

$$e_1 = (1, 0, \cdots, 0), e_2 = (0, 1, \cdots, 0), \cdots, e_r = (0, \cdots, 0, 1),$$

which clearly form a constant frame for $U \times \mathbb{C}^r$, and thus $\{(h_*)^{-1}(e_1), \cdots, (h_*)^{-1}(e_r)\}$ forms a frame for $E|_U$, since the bundle mapping h is an isomorphism on fibres, carrying a basis to a basis. $\square$

Remark. We see that having a frame is equivalent to having a trivialization and that the existence of a global frame defined over X is equivalent to the bundle being trivial.

Suppose that $E \to X$ is a vector bundle of rank r and that $f^T = (s_1, \cdots, s_r)$ is a frame at $x \in X$; i.e., there is a neighborhood U of x and sections $\{s_1, \cdots, s_r\}$, $s_j \in \Gamma(U, E)$, which are linearly independent at each point of U.

Let $\psi : U \to \mathrm{GL}(r, \mathbb{C})$ is a differentiable mapping, i.e. $\psi(x) = (\psi_{ij}(x))$, where $\psi_{ij}(x)$ is a $\mathbb{C}$-valued differentiable map for all $x \in U$. Then there is an action of ψ on the set of all frames on the open set U defined by

$$f \mapsto \psi \cdot f,$$

where

$$(\psi \cdot f)(x) = \left(\sum_{i=1}^{r} \psi_{1i}(x) s_i(x), \cdots, \sum_{i=1}^{r} \psi_{ri}(x) s_i(x) \right)^T, \quad x \in U,$$

is also a frame. Clearly, $(\psi \cdot f)(x) = \psi(x) \cdot f(x)$, where we use the usual matrix product.

Definition 1.3. *The above map $\psi : U \to \mathrm{GL}(r, \mathbb{C})$ is called a* ***change of frame****.*

Remark. Given any two frames f and f' over U, we see that there exists a change of frame ψ defined over U such that $f' = \psi \cdot f$.

Let $f^T = (s_1, \cdots, s_r)$ be a frame over U for E and $\xi \in \Gamma(U, E)$. Then

$$\xi = \sum_{i=1}^{r} \xi^i(f) s_i,$$

where $\xi^i(f) \in C^\infty(U)$ are uniquely determined smooth functions on U. This induces a map

$$\Gamma(U, E) \xrightarrow{\sim} C^\infty(U)^r \cong \Gamma(U, U \times \mathbb{C}^r)$$

by

$$\xi \mapsto \xi(f) = \begin{bmatrix} \xi^1(f) \\ \vdots \\ \xi^r(f) \end{bmatrix}.$$

Proposition 1.4. *Suppose that $f^T = (s_1, \cdots, s_r)$ is a frame over U and ψ is a change of frame over U. Then $\xi(\psi \cdot f) = (\psi^T)^{-1} \cdot \xi(f)$.*

Proof. We see that

$$\xi = \sum_{i=1}^{r} \xi^i(\psi \cdot f) \left(\sum_{j=1}^{r} \psi_{ij} s_j \right) = \sum_{j=1}^{r} \sum_{i=1}^{r} \xi^i(\psi \cdot f) \psi_{ij} s_j.$$

Compared with $\xi = \sum_{j=1}^{r} \xi^j(f) s_j$, we see that

$$\xi^j(f) = \sum_{i=1}^{r} \xi^i(\psi \cdot f) \psi_{ij}$$

for all j. Equivalently,

$$\xi(f) = \psi^T \cdot \xi(\psi \cdot f)$$

or

$$\xi(\psi \cdot f) = (\psi^T)^{-1} \cdot \xi(f).$$

$\square$

If E is a holomorphic vector bundle, then we can define the **holomorphic frames** similarly, i.e., $f^T = (s_1, \cdots, s_r)$, $s_j \in \mathcal{O}_X(U, E)$, and $s_1 \wedge \cdots \wedge s_r(x) \neq 0$, for $x \in$; and **holomorphic changes of frame**, i.e., holomorphic mappings $\psi : U \to \mathrm{GL}(r, \mathbb{C})$. Correspondingly, if $\xi \in \mathcal{O}_X(U, E)$, then $\xi(f) \in \mathcal{O}_X(U)^r$.

Definition 1.5. *A* ***Hermitian metric*** *or* ***Hermitian structure*** *h on a vector bundle $E \to X$ is a smooth field of Hermitian inner products on the fibers of E, that is, for every $x \in X$,*

$$h_x : E_x \times E_x \to \mathbb{C}$$

satisfies

(1) $h_x(u, v)$ *is $\mathbb{C}$–linear in u for every $v \in E_x$.*
(2) $h_x(u, v) = \overline{H_x(v, u)}, \ \forall u, v \in E_x$.
(3) $h_x(u, u) > 0, \ \forall u \neq 0$.
(4) $h_x(u, v)$ *is a smooth function on X for every smooth sections u, v of E.*

Remark. It is clear from the above conditions that h is $\mathbb{C}$–antilinear in the second variable. The third condition shows that h is non–degenerate. In fact, it is quite useful to think to h as to a $\mathbb{C}$–antilinear isomorphism $h : E \to E^*$.

Moreover, we see that $h_x(iu, iv) = h_x(u, v)$ for all $u, v \in E_x$.

Definition 1.6. *A vector bundle E equipped with a Hermitian metric h is called a* ***Hermitian vector bundle****.*

Suppose that $E \to X$ is a Hermitian vector bundle and that $f = (s_1, \cdots, s_r)$ is a frame for $E \to X$ over some open set U. Define

$$h(f)_{ij} = h(s_i, s_j)$$

and let

$$h(f) = [h(f)_{ij}]$$

be the $r \times r$ matrix of the C^∞ functions $\{h(f)_{ij}\}$, where $r = \operatorname{rank} E$. We see that $h(f)$ is a positive definite Hermitian symmetric matrix and is a local representative for the Hermitian metric h with respect to the frame f.

Theorem 1.7. *Every rank r complex vector bundle $E \to X$ admits a Hermitian metric.*

1.2 Connections

Let X be a real manifold and $\pi : E \to X$ be a complex vector bundle on X. We denote by $\mathcal{A}^i(X, E)$ the sheaf of i-forms with values in E, i.e.

$$\mathcal{A}^i(X, E) = \Gamma\left(X, \bigwedge^i (T^*X) \otimes E\right),$$

where we adopt the notation $\otimes := \otimes_{C^\infty(X)}$. Let $\operatorname{End}(E)$ be the $C^\infty(X)$-endomorphisms bundle of E, i.e. $\operatorname{End}(E) = \operatorname{Hom}_{C^\infty(X)}(E, E) = E^* \otimes_{C^\infty(X)} E$.

Definition 1.8. *A **connection** on a vector bundle E is a $\mathbb{C}$-linear sheaf morphism*

$$\nabla : \mathcal{A}^0(X, E) \to \mathcal{A}^1(X, E)$$

which satisfies the Leibniz rule

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s)$$

for any function f on M and local section s of E.

Definition 1.9. *A section s of a vector bundle E is called **parallel** or **flat** with respect to a connection ∇ on E if $\nabla(s) = 0$.*

Proposition 1.10. *If ∇ and ∇' are two connections on a vector bundle E, then $\nabla - \nabla'$ is $C^\infty(X)$-linear.*

In particular, $\nabla - \nabla' \in \mathcal{A}^1(X, \operatorname{End}(E))$.

Proof. For any $f \in C^\infty(M)$, by Leibniz rule, we have

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s)$$

and

$$\nabla'(f \cdot s) = df \otimes s + f \cdot \nabla'(s).$$

Thus,

$$(\nabla - \nabla')(f \cdot s) = f \cdot (\nabla - \nabla')(s).$$

Note that $\nabla - \nabla'$ can be identified with a global section of

$$\operatorname{Hom}_{C^\infty(X)}\left(E, \bigwedge^1 T^*X \otimes E\right) \cong E^* \otimes_{C^\infty(M)} \bigwedge^1 T^*X \otimes E \cong \bigwedge^1 T^*X \otimes \operatorname{End}_{C^\infty(X)}(E).$$

So, $\nabla - \nabla' \in \mathcal{A}^1(X, \operatorname{End}(E))$. ∎

Proposition 1.11. *If ∇ is a connection on E and $a \in \mathcal{A}^1(M, \mathrm{End}(E))$, then $\nabla + a$ is again a connection on E.*

Proof. For any $f \in \mathcal{A}^0(M)$, we have $a(s \cdot s) = f \cdot a(s)$. Thus, we see that $(\nabla + a)(f \cdot s) = \nabla(f \cdot s) + a(f \cdot s) = df \otimes s + f \cdot \nabla(s) + f \cdot a(s) = df \otimes s + f \cdot (\nabla + a)(s)$. So, $\nabla + a$ is a connection on E. □

By the definition of affine space over a vector space, we have:

Corollary 1.12. *The set of all connections on a vector bundle E is an affine space over the complex vector space $\mathcal{A}^1(M, \mathrm{End}(E))$ in a natural way.*

We now give a local description of a connection ∇ on a vector bundle $E \to X$.

Definition 1.13. *Let f be a frame over U for a vector bundle $E \to X$, equipped with a connection ∇. We define the **connection matrix** $\omega(\nabla, f)$ by setting*

$$\omega(\nabla, f) = (\omega_{ij}(\nabla, f)),$$

where $\omega_{ij}(\nabla, f)$ are complex-valued 1-forms in U with

$$\nabla(s_j) = \sum_{i=1}^{r} \omega_{ij}(\nabla, f) \otimes s_i. \tag{1.2.1}$$

We abuse the notation $\omega(f) = \omega(\nabla, f)$ and $\omega_{ij}(f) = \omega_{ij}(\nabla, f)$ when there is no danger of confusion.

Remark. The equation (1.3.3) can be written as

$$\nabla \cdot f = \omega \cdot f.$$

Note that ∇ is not $C^\infty(X)$-linear, so we don't have $(\nabla\xi)(f) = \omega(f) \cdot \xi(f)$ in general.

Proposition 1.14. *Let U be an open subset of X, and let $f^T = (s_1, \cdots, s_r)$ be a frame over U. Then, locally we have*

$$\nabla = d + \omega(f),$$

where $\omega(f)$ is the connection matrix of ∇ with respect to f, in the sense $(\nabla\xi)(f) = [d + \omega(f)]\xi(f)$.

Proof. For an arbitrary section ξ of E over U, we can write it as

$$\xi = \sum_i \xi^i(f) s_i, \tag{1.2.2}$$

where $\xi^i(f)$ are complex-valued C^∞-functions in U. Then we have

$$\begin{aligned}\nabla\xi &= \sum_j \nabla(\xi^j(f)s_j)\\ &= \sum_j \left(d\xi^j(f)\otimes s_j + \xi^j(f)\nabla(s_j)\right)\\ &= \sum_j \left(d\xi^j(f)\otimes s_j + \xi^j(f)\sum_i \omega_{ij}(f)\otimes s_i\right)\\ &= \sum_j d\xi^j(f)\otimes s_j + \sum_j\left(\sum_k \omega_{jk}(f)\xi^k(f)\right)\otimes s_j.\end{aligned}$$

Thus, we see that

$$(\nabla\xi)(f) = d\xi(f) + \omega(f)\xi(f) = [d+\omega(f)]\xi(f).$$

Thus, we have $\nabla = d + \omega(f)$, where we have set

$$d\xi(f) = \begin{bmatrix} d\xi^1(f)\\ \vdots \\ d\xi^r(f)\end{bmatrix},$$

by thinking of $d+\omega(f)$ as being an operator acting on vector-valued functions. □

Example 1.15. *Let E_1 and E_2 be two vector bundles on M endowed with connections ∇_1 and ∇_2.*

(1) *If s_1 and s_2 are local sections of E_1 and E_2, we set*

$$\nabla(s_1\oplus s_2) = \nabla_1(s_1)\oplus\nabla_2(s_2).$$

This defines a natural connection on the direct sum $E_1\oplus E_2$.

(2) *If s_1 and s_2 are local sections of E_1 and E_2, we set*

$$\nabla(s_1\otimes s_2) = \nabla_1(s_1)\otimes s_2 + s_1\otimes\nabla_2(s_2).$$

This defines a natural connection on the tensor product $E_1\otimes E_2$. It is routine to check that ∇ is well-defined and indeed a connection.

(3) *Let $f : E_1 \to E_2$ be a morphism of vector bundles, i.e. f is a section on* $\mathrm{Hom}(E_1, E_2)$*. Let s_1 be a local section of E_1, then $f(s_1) = f\circ s_1$ is a local section on E_2. A natural connection*

$$\nabla^H : \mathcal{A}^0(X, \mathrm{Hom}(E_1,E_2)) \to \mathcal{A}^1(X, \mathrm{Hom}(E_1,E_2))$$

on $\mathrm{Hom}(E_1,E_2)$ *can be defined by*

$$f \mapsto \nabla^H f,$$

where

$$(\nabla^H f)(s_1) = \nabla_2(f(s_1)) - f(\nabla_1(s_1)).$$

In the second term, f is applied to $\nabla_1(s_1)$ according to $f(\alpha\otimes t) = \alpha\otimes f(t)$ for $\alpha\in\mathcal{A}^1(X,E)$ and $t\in\mathcal{A}^0(X,E)$.

Note that $\nabla^H f$ sends a section of E_1 to a section of $T^*X \otimes E_2$, we can consider ∇^H as a morphism $\nabla^H f : E_1 \to T^*X \otimes E_2$ of vector bundles. So, $\nabla^H f \in \Gamma(\mathrm{Hom}(E_1, T^*X \otimes E_2)) \cong \Gamma(T^*X \otimes E_1^* \otimes E_2) \cong \Gamma(T^*X \otimes \mathrm{Hom}(E_1, E_2)) = \mathcal{A}^1(X, \mathrm{Hom}(E_1, E_2))$.

(4) *Let E be a vector bundle equipped with a connection ∇. Take $E_1 = E$ and $E_2 = X \times \mathbb{C}$ to be the trivial bundle with trivial connection d. Then we have a connection ∇^* on the dual bundle E^* by*

$$\nabla^*(f)(s) = d(f(s)) - f(\nabla(s)).$$

(5) *Let $f : M \to N$ be a differentiable map and let ∇ be a connection on a vector bundle E over N. Let ∇ over an open subset $U_i \subset N$ be of the form $d + \omega_i$ (after trivializing $E|_{U_i}$). Then the pull-back connection $f^*\nabla$ on the pull-back vector bundle f^*E over M is locally defined by*

$$f^*\nabla|_{f^{-1}(U_i)} = d + f^*\omega_i.$$

It is straightforward to see that the locally given connections glue to a global one on f^*E.

Definition 1.16. *Let (E, h) be an Hermitian vector bundle. A connection ∇ on E is an **Hermitian connection** with respect to h if for any local sections s_1, s_2 one has*

$$d(h(s_1, s_2)) = h(\nabla(s_1), s_2) + h(s_1, \nabla(s_2)).$$

Lemma 1.17. *Let (E, h) be an Hermitian vector bundle. A connection ∇ on E is an Hermitian connection with respect to h if and only if*

$$dh(f) = \omega(f)^T \cdot h(f) + h(f) \cdot \overline{\omega(f)}$$

for all frames $f = (s_1, \cdots, s_r)^T$.

Proof. First, let $f = (s_1, \cdots, s_r)^T$ be any frame and that ∇ an Hermitian connection with respect to h on E. Then we see that

$$\begin{aligned} dh(f)_{ij} &= dh(s_i, s_j) \\ &= h(\nabla(s_i), s_j) + h(s_i, \nabla(s_j)) \\ &= h\left(\sum_{k=1}^{r} \omega_{ki}(f) \otimes s_k, s_j\right) + h\left(s_i, \sum_{\ell=1}^{r} \omega_{\ell j}(f) \otimes s_\ell\right) \\ &= \sum_{k=1}^{r} \omega_{ki}(f) h(s_k, s_j) + \sum_{\ell=1}^{r} \overline{\omega_{\ell j}(f)} h(s_i, s_\ell) \\ &= \sum_{k=1}^{r} \omega_{ki}(f) h(f)_{kj} + \sum_{\ell=1}^{r} \overline{\omega_{\ell j}(f)} h(f)_{i\ell} \end{aligned}$$

So, we have

$$dh(f) = \omega(f)^T \cdot h(f) + h(f) \cdot \overline{\omega(f)}.$$

Conversely, suppose $dh(f) = \omega(f)^T \cdot h(f) + h(f) \cdot \overline{\omega(f)}$ is satisfied for all frames f. Then, in terms of a local frame, one obtains immediately

$$dh(\xi,\eta) = dh\left(\sum_{i=1}^{r}\xi^i(f)s_i, \sum_{j=1}^{r}\eta^j(f)s_j\right) = d\left(\sum_{i=1}^{r}\sum_{j=1}^{r}\xi^i(f)\overline{\eta^j(f)}h(s_i,s_j)\right)$$
$$= d\left(\sum_{i=1}^{r}\sum_{j=1}^{r}\xi^i(f)\overline{\eta^j(f)}h(f)_{ij}\right) = d\left(\overline{\eta(f)}^T h(f)^T\xi(f)\right)$$
$$= \left(d\overline{\eta(f)}\right)^T h(f)^T\xi(f) + \overline{\eta(f)}^T(dh(f))^T\xi(f) + \overline{\eta(f)}^T h(f)^T d\xi(f)$$

$$\begin{aligned}
&= \left(d\overline{\eta(f)}\right)^T h(f)^T\xi(f) + \overline{\eta(f)}^T[h(f)^T\cdot\omega(f) + \overline{\omega(f)}^T\cdot h(f)^T]\xi(f) + \overline{\eta(f)}^T h(f)^T d\xi(f)\\
&= \left(d\overline{\eta(f)}\right)^T h(f)^T\xi(f) + \overline{\eta(f)}^T h(f)^T\omega(f)\xi(f) + \overline{\eta(f)}^T\overline{\omega(f)}^T h(f)^T\xi(f) + \overline{\eta(f)}^T h(f)^T d\xi(f)\\
&= \left(d\overline{\eta(f)} + \overline{\omega(f)\eta(f)}\right)^T h(f)^T\xi(f) + \overline{\eta(f)}^T h(f)^T\left(d\xi(f) + \omega(f)\xi(f)\right)\\
&= \overline{(d+\omega(f))\,\eta(f)}^T h(f)^T\xi(f) + \overline{\eta(f)}^T h(f)^T\left(d+\omega(f)\right)\xi(f)\\
&= h(\xi,\nabla\eta) + h(\nabla\xi,\eta).
\end{aligned}$$

□

Definition 1.18. *A frame f is called **unitary** if $h(f) = I$.*

Lemma 1.19. *Unitary frames always exists near a given point $x_0 \in U$.*

Proof. The Gram-Schmidt orthogonalization process allows one to find r local sections which form an orthonormal basis for E_x at all points x near x_0. □

Lemma 1.20. *Let ψ be a change of frame, then*

$$d\psi^T + \omega(f)\cdot\psi^T = \psi^T\cdot\omega(\psi\cdot f).$$

Proof. Suppose $\psi\cdot f = \left(\sum_i\psi_{1i}s_i,\cdots,\sum_i\psi_{ri}s_i\right)^T = (e_1,\cdots,e_r)^T$. Then,

$$\begin{aligned}
\nabla(e_j) &= \sum_i\omega_{ij}(\psi\cdot f)\otimes e_i\\
&= \sum_i\omega_{ij}(\psi\cdot f)\otimes\left(\sum_k\psi_{ik}s_k\right)\\
&= \sum_i\sum_k\omega_{ij}(\psi\cdot f)\psi_{ik}\otimes s_k\\
&= \sum_k\left(\sum_i\omega_{ij}(\psi\cdot f)\psi_{ik}\right)\otimes s_k
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\nabla\left(\sum_k\psi_{jk}s_k\right) &= \sum_k d\psi_{jk}\otimes s_k + \sum_k\sum_\ell\omega_{k\ell}(f)\psi_{j\ell}\otimes s_k\\
&= \sum_k\left(d\psi_{jk} + \sum_\ell\omega_{k\ell}(f)\psi_{j\ell}\right)\otimes s_k.
\end{aligned}$$

By comparing coefficients, we obtain

$$\sum_i \omega_{ij}(\psi \cdot f)\psi_{ik} = d\psi_{jk} + \sum_\ell \omega_{k\ell}(f)\psi_{j\ell}$$

for each j, k. It follows that

$$\psi^T \cdot \omega(\psi \cdot f) = d\psi^T + \omega(f) \cdot \psi^T.$$

□

Proposition 1.21. *Let $E \to X$ be a Hermitian vector bundle. Then there exists an Hermitian connection ∇ on E with respect to the Hermitian metric h on E.*

Proof. We can find a locally finite covering U_α and unitary frames f_α defined in U_α. Then

$$dh(f) = \omega(f)^T \cdot h(f) + h(f) \cdot \overline{\omega(f)}$$

becomes

$$0 = \omega(f)^T + \overline{\omega(f)}$$

for a unitary frame; i.e., $\omega(f_\alpha)$ is to be skew-Hermitian.

In each U_α we can choose the trivial skew-Hermitian matrix of the form $\omega_\alpha = 0$; i.e., $\omega(f_\alpha) = 0$. If we make a change of frame in U_α, then we see that we require that

$$\omega(\psi \cdot f_\alpha) = (\psi^T)^{-1} d\psi^T$$

by Lemma 1.20. Therefore, define $\omega(\psi \cdot f_\alpha)$ by $(\psi^T)^{-1} d\psi^T$, then we see that

$$\begin{aligned}
h(\psi \cdot f_\alpha)_{k\ell} = h(e_k, e_\ell) &= h\left(\sum_i \psi_{ki} s_i, \sum_j \psi_{\ell j} s_j\right) \\
&= \sum_i \sum_j \psi_{ki} \overline{\psi_{\ell j}} h(s_i, s_j) \\
&= \sum_i \sum_j \psi_{ki} \overline{\psi_{\ell j}} h(f_\alpha)_{ij} \\
&= \left(\psi h(f_\alpha) \overline{\psi}^T\right)_{k\ell}.
\end{aligned}$$

So,

$$h(\psi \cdot f_\alpha) = \psi h(f_\alpha) \overline{\psi}^T = \psi \cdot \overline{\psi}^T.$$

It follows that

$$\begin{aligned}
dh(\psi \cdot f_\alpha) &= d(\psi \cdot \overline{\psi}^T) \\
&= d\psi \cdot \overline{\psi}^T + \psi \cdot \overline{d\psi}^T \\
&= d\psi \cdot \psi^{-1} \cdot \psi \cdot \overline{\psi}^T + \psi \cdot \overline{\psi}^T \cdot (\overline{\psi}^T)^{-1} \cdot \overline{d\psi}^T \\
&= \omega(\psi \cdot f_\alpha)^T \cdot h(\psi \cdot f_\alpha) + h(\psi \cdot f_\alpha) \cdot \overline{\omega(\psi \cdot f_\alpha)},
\end{aligned}$$

which verifies the compatibility.

Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$ and let D_α be the connection in $E|_{U_\alpha}$ defined by

$$(D_\alpha \xi)(f_\alpha) = d\xi(f_\alpha),$$

in which cases $\omega(D_\alpha, f_\alpha) = 0$. In general, D_α is defined with respect to other frames over U_α by

$$(D_\alpha \xi)(\psi \cdot f_\alpha) = d\xi(\psi \cdot f_\alpha) + \omega(D_\alpha, \psi \cdot f_\alpha)\xi(\psi \cdot f_\alpha),$$

where $\omega(D_\alpha, \psi \cdot f_\alpha) = (\psi^T)^{-1} d\psi^T$. By the above discussion, we see that D_α is an Hermitian connection with respect to h on E. Now, let $D = \sum_\alpha \varphi_\alpha D_\alpha$, which is an Hermitian connection with respect to h on E as

$$h(D\xi, \eta) + h(\xi, D\eta) = \sum_\alpha \varphi_\alpha [h(D_\alpha \xi, \eta) + h(\xi, D_\alpha \eta)] = \sum_\alpha \varphi_\alpha dh(\xi, \eta) = dh(\xi, \eta).$$

□

1.3 Curvature

Proposition 1.22. *The connection has a natural extension to an operator*

$$\nabla : \mathcal{A}^k(X, E) \to \mathcal{A}^{k+1}(X, E)$$

uniquely defined by

(a) $\nabla|_{\mathcal{A}^0(X,E)} = \nabla$.

(b) $\forall \omega \in \mathcal{A}^k(X)$, $\eta \in \mathcal{A}^0(X, E)$, *we have*

$$\nabla(\omega \otimes \eta) = d\omega \otimes \eta + (-1)^k \omega \wedge \nabla \eta.$$

Proof. We first show the existence. Let $\{U_\alpha\}$ be a covering realizing the local trivialization $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^r$ and $\{\varphi_\alpha\}_{\alpha \in A}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then,

$$\eta = \sum_{\alpha \in A} \varphi_\alpha \eta,$$

which is a locally finite sum as $\{\varphi_\alpha^{-1}((0, 1]) | \alpha \in A\}$ is locally finite. Over U_α, we can define ∇ by

$$\nabla(\omega_\alpha \otimes \eta_\alpha) = d\omega_\alpha \otimes \eta_\alpha + (-1)^k \omega_\alpha \wedge \nabla \eta_\alpha.$$

This is well-defined as for any $C^\infty(U_\alpha)$ function f_α, we have

$$\begin{aligned}
\nabla(\omega_\alpha \otimes (f_\alpha \cdot \eta_\alpha)) &= d\omega_\alpha \otimes (f_\alpha \cdot \eta_\alpha) + (-1)^k \omega_\alpha \wedge \nabla(f_\alpha \cdot \eta_\alpha) \\
&= f_\alpha \cdot d\omega_\alpha \otimes \eta_\alpha + (-1)^k \omega_\alpha \wedge (df_\alpha \otimes \eta_\alpha + f_\alpha \nabla \eta_\alpha) \\
&= f_\alpha \cdot d\omega_\alpha \otimes \eta_\alpha + (-1)^k \omega_\alpha \wedge df_\alpha \otimes \eta_\alpha + (-1)^k \omega_\alpha \wedge f_\alpha \nabla \eta_\alpha \\
&= (f_\alpha \cdot d\omega_\alpha + df_\alpha \wedge \omega_\alpha) \otimes \eta_\alpha + (-1)^k f_\alpha \omega_\alpha \wedge \nabla \eta_\alpha \\
&= d(f_\alpha \omega_\alpha) \otimes \eta_\alpha + (-1)^k f_\alpha \omega_\alpha \wedge \nabla \eta_\alpha \\
&= \nabla(f_\alpha \omega_\alpha \otimes \eta_\alpha).
\end{aligned}$$

Then, globally, we have

$$\begin{aligned}\nabla(\omega\otimes\eta) &= \nabla\left(\sum_\alpha \varphi_\alpha\omega\otimes\sum_\beta\varphi_\beta\eta\right)\\ &= \sum_\alpha\sum_\beta \nabla\left(\varphi_\alpha\omega\otimes\varphi_\beta\eta\right)\\ &= \sum_\alpha\sum_\beta\left[d(\varphi_\alpha\omega)\otimes(\varphi_\beta\eta)+(-1)^k\varphi_\alpha\omega\wedge\nabla(\varphi_\beta\eta)\right]\\ &= d\omega\otimes\eta+(-1)^k\omega\wedge\nabla\eta.\end{aligned}$$

The uniqueness is clear. $\square$

Proposition 1.23. *The extension ∇ satisfies the generalized Leibniz rule, i.e. $\forall\alpha\in\mathcal{A}^r(X)$, $\beta\in\mathcal{A}^s(X,E)$, we have*

$$\nabla(\alpha\wedge\beta)=d\alpha\wedge\beta+(-1)^r\alpha\wedge\nabla\beta.$$

Proof. By the $\mathbb{C}$-linearity of ∇, we may assume that $\beta = t\otimes s$, where $t\in\mathcal{A}^s(X)$ and $s\in\mathcal{A}^0(X,E)$. Thus,

$$\begin{aligned}\nabla(\alpha\wedge\beta) &= \nabla((\alpha\wedge t)\otimes s)\\ &= d(\alpha\wedge t)\otimes s+(-1)^{r+s}(\alpha\wedge t)\otimes\nabla(s)\\ &= d\alpha\wedge t\otimes s+(-1)^r\alpha\wedge dt\otimes s+(-1)^{r+s}\alpha\wedge t\otimes\nabla s\\ &= d\alpha\wedge\beta+(-1)^r\alpha\wedge(dt\otimes s+(-1)^s t\otimes\nabla s)\\ &= d\alpha\wedge\beta+(-1)^r\alpha\wedge\nabla(t\otimes s)\\ &= d\alpha\wedge\beta+(-1)^r\alpha\wedge\nabla\beta.\end{aligned}$$

$\square$

Definition 1.24. *Let E be a vector bundle with a connection ∇ on E. The* ***curvature***

$$F_\nabla:\mathcal{A}^0(X,E)\to\mathcal{A}^2(X,E)$$

of ∇ is the composition

$$F_\nabla := \nabla\circ\nabla$$

i.e.

$$F_\nabla:\mathcal{A}^0(X,E)\xrightarrow{\nabla}\mathcal{A}^1(X,E)\xrightarrow{\nabla}\mathcal{A}^2(X,E)$$

Proposition 1.25. *The curvature morphism $F_\nabla:\mathcal{A}^0(X,E)\to\mathcal{A}^2(X,E)$ is $C^\infty(X)$-linear.*

Proof. For any $f\in C^\infty(M)$, we have

$$\begin{aligned}F_\nabla(f\cdot s) &= \nabla(\nabla(f\cdot s))\\ &= \nabla(df\otimes s+f\cdot\nabla(s))\\ &= d^2(f)\otimes s-df\wedge\nabla(s)+df\wedge\nabla(s)+f\cdot\nabla(\nabla(s))\\ &= f\cdot F_\nabla(s).\end{aligned}$$

$\square$

By this proposition, we can consider F_∇ as a global section of $\bigwedge^2 T^*X \otimes \mathrm{End}(E)$. Indeed, F_∇ can be identified with a global section of

$$\mathrm{Hom}_{C^\infty(X)}\left(E, \bigwedge^2 T^*X \otimes E\right) \cong E^* \otimes_{C^\infty(M)} \bigwedge^2 T^*X \otimes E \cong \bigwedge^2 T^*X \otimes \mathrm{End}_{C^\infty(X)}(E).$$

So, $F_\nabla \in \mathcal{A}^2(X, \mathrm{End}(E))$.

Recall that we identify $\mathrm{Hom}(V, W) \cong V^* \otimes W$ by using the following map

$$V^* \otimes W \to \mathrm{Hom}(V, W)$$

$$v^* \otimes w \mapsto (x \mapsto v(x)w),$$

whose inverse map is given by

$$\mathrm{Hom}(V, W) \to V^* \otimes W$$

$$f \mapsto \sum_{i=1}^{r} e_i^* \otimes f(e_i),$$

where $\{e_1, \cdots, e_r\}$ is a basis for V, and $\{e_1^*, \cdots, e_r^*\}$ is the dual basis for V^*.

If $\{s_1, \cdots, s_r\}$ is a frame on E and $\{s_1^*, \cdots, s_r^*\}$ is the dual frame on E^*, we see that

$$F_\nabla = \sum_{j=1}^{r} F_\nabla(s_j) \otimes s_j^* \tag{1.3.1}$$

as an element in $\bigwedge^2 T^*X \otimes E \otimes E^*$.

We define a bilinear map $\wedge$ for any vector bundle E:

$$\wedge : \mathcal{A}^r(X, \mathrm{End}(E)) \times \mathcal{A}^s(X, \mathrm{End}(E)) \to \mathcal{A}^{r+s}(X, E),$$

uniquely determined by

$$(\omega^r \otimes A) \wedge (\eta^s \otimes B) \mapsto \omega^r \wedge \eta^s \otimes AB,$$

where $\omega^r \in \mathcal{A}^r(X), \eta^s \in \mathcal{A}^s(X)$ and $A, B \in C^\infty(\mathrm{End}(E))$.

So, if $A = (\alpha_{ij}^r)$ and $B = (\beta_{ij}^s)$ for $\alpha_{ij}^r \in \mathcal{A}^r(X), \beta_{ij}^s \in \mathcal{A}^s(X)$, then $A \wedge B = (c_{ij}^{r+s})$, where $c_{ij}^{r+s} = \sum_k \alpha_{ik}^r \wedge \beta_{kj}^s$.

Proposition 1.26. *Let U be an open subset of X, and $f^T = (s_1, \cdots, s_r)$ a frame over U. Let $\omega(f)$ is the connection matrix of $\nabla : \mathcal{A}^0(X, E) \to \mathcal{A}^1(X, E)$ with respect to f.*

Then, for any $\nabla : \mathcal{A}^k(X, E) \to \mathcal{A}^{k+1}(X, E)$, we have

$$\nabla = d + \omega(f)$$

locally on U in the sense $(\nabla\xi)(f) = [d + \omega(f)]\xi(f)$, where $\xi \in \mathcal{A}^k(X, E)$.

Proof. For an arbitrary $\xi \in \mathcal{A}^k(U, E)$, we can write it as

$$\xi = \sum_i \xi^i(f) \otimes s_i, \tag{1.3.2}$$

where $\xi^i(f)$ are complex-valued C^∞-differential k-forms in U. Then we have

$$\begin{aligned}
\nabla\xi &= \sum_j \nabla(\xi^j(f)\otimes s_j)\\
&= \sum_j \big(d\xi^j(f)\otimes s_j + (-1)^k\xi^j(f)\wedge\nabla(s_j)\big)\\
&= \sum_j \left(d\xi^j(f)\otimes s_j + (-1)^k\xi^j(f)\wedge\left(\sum_i \omega_{ij}(f)\otimes s_i\right)\right)\\
&= \sum_j \left(d\xi^j(f)\otimes s_j + (-1)^k\left(\sum_i \xi^j(f)\wedge\omega_{ij}(f)\otimes s_i\right)\right)\\
&= \sum_j d\xi^j(f)\otimes s_j + \sum_j\left(\sum_k \omega_{jk}(f)\wedge\xi^k(f)\right)\otimes s_j.
\end{aligned}$$

Thus, we see that

$$(\nabla\xi)(f) = d\xi(f) + \omega(f)\wedge\xi(f) = [d+\omega(f)]\xi(f).$$

Thus, we have $\nabla = d+\omega(f)$, where we have set

$$d\xi(f) = \begin{bmatrix} d\xi^1(f)\\ \vdots \\ d\xi^r(f)\end{bmatrix},$$

by thinking of $d+\omega(f)$ as being an operator acting on vector-valued diffrential k-forms. $\square$

Definition 1.27. *Let U be an open subset of X, and let $f^T = (s_1,\cdots,s_r)$ be a frame over U. We define the **curvature matrix** $\Omega(\nabla,f)$ by setting*

$$\Omega(\nabla,f) = (\Omega_{ij}(\nabla,f)),$$

where $\Omega_{ij}(\nabla,f)$ are complex-valued 2-forms in U with

$$F_\nabla(s_j) = \sum_{i=1}^r \Omega_{ij}(\nabla,f)\otimes s_i. \tag{1.3.3}$$

We abuse the notation $\Omega(f) = \Omega(\nabla,f)$ and $\Omega_{ij}(f) = \Omega_{ij}(\nabla,f)$ when there is no danger of confusion.

Proposition 1.28. *Let U be an open subset of X, and let $f^T = (s_1,\cdots,s_r)$ be a frame over U. Then, we have*

$$\Omega(f) = d\omega(f) + \omega(f)\wedge\omega(f)$$

locally on U, where $\omega(f)$ is the connection matrix of $\nabla : \mathcal{A}^0(X,E)\to\mathcal{A}^1(X,E)$ with respect to f.

Proof. Let ξ be a section of E, i.e. $\xi\in\mathcal{A}^0(X,E)$. Recall that F_∇ is $C^\infty(X)$-linear, we have

$$(F_\nabla\xi)(f) = \Omega(f)\cdot\xi(f).$$

Then,

$$\begin{aligned}
\Omega(f)\cdot\xi(f) &= (F_\nabla\xi)(f) = (\nabla(\nabla\xi))(f)\\
&= [d+\omega(f)](\nabla\xi)(f)\\
&= [d+\omega(f)]\circ[d+\omega(f)]\xi(f)\\
&= [d+\omega(f)][d\xi(f)+\omega(f)\xi(f)]\\
&= d^2\xi(f)+\omega(f)\wedge d\xi(f)+d[\omega(f)\xi(f)]+\omega(f)\wedge[\omega(f)\xi(f)]\\
&= \omega(f)\wedge d\xi(f)+[d\omega(f)]\cdot\xi(f)+(-1)\omega(f)\wedge d\xi(f)+[\omega(f)\wedge\omega(f)]\xi(f)\\
&= [d\omega(f)+\omega(f)\wedge\omega(f)]\cdot\xi(f).
\end{aligned}$$

So,

$$\Omega(f) = d\omega(f)+\omega(f)\wedge\omega(f).$$

□

Corollary 1.29 (Bianchi identity).

$$d\Omega(f) = \Omega(f)\wedge\omega(f) - \omega(f)\Omega(f) = [\Omega(f),\omega(f)].$$

Proof.

$$\begin{aligned}
d\Omega(f) &= d^2\omega(f)+d[\omega(f)\wedge\omega(f)]\\
&= d\omega(f)\wedge\omega(f)-\omega(f)\wedge d\omega(f)\\
&= d\omega(f)\wedge\omega(f)+\omega(f)\wedge\omega(f)\wedge\omega(f)\\
&\qquad\qquad -\omega(f)\wedge\omega(f)\wedge\omega(f)-\omega(f)\wedge d\omega(f)\\
&= \Omega(f)\wedge\omega(f)-\omega(f)\wedge\Omega(f).
\end{aligned}$$

□

Let E be a vector bundle with a connection ∇. By Example 1.15(3), we see that ∇ induces a natural connection $\nabla^{\mathrm{End}(E)}$ on $\mathrm{End}(E)$. This extends to a operator

$$\nabla^{\mathrm{End}(E)} : \mathcal{A}^k(X,\mathrm{End}(E)) \to \mathcal{A}^{k+1}(X,\mathrm{End}(E)).$$

Now, since the curvature F_∇ of the connection ∇ can be regarded as an element on $\mathcal{A}^2(X,\mathrm{End}(E))$, the notation $\nabla^{\mathrm{End}(E)}(F_\nabla)\in\mathcal{A}^3(X,\mathrm{End}(E))$ makes sense. The Bianchi identity states that

$$\nabla^{\mathrm{End}(E)}(F_\nabla) = 0.$$

Before proving this result, we need the following lemmas.

Lemma 1.30. *Let E be a vector bundle with a connection ∇ and $f=(s_1,\cdots,s_r)^T$ a frame over U on E and $f^*=(s_1^*,\cdots,s_r^*)^T$ be the dual frame on the dual bundle E^*, i.e.*

$$s_i^*(s_j) = \delta_{ij}.$$

Let ∇^* be the natural connection on E^* induced by ∇. Then, we have

$$\omega(\nabla^*,f^*) = -\omega(\nabla,f)^T.$$

Proof. Recall that

$$\nabla s_i = \sum_{\ell=1}^{r} \omega_{\ell i}(f) \otimes s_\ell$$

and

$$\begin{aligned}(\nabla^* s_j^*)(s_i) &= d(s_j^*(s_i)) - s_j^*(\nabla s_i) = d(\delta_{ij}) - s_j^* \left(\sum_{\ell=1}^{r} \omega_{\ell i}(f) \otimes s_\ell \right) \\ &= -\sum_{\ell=1}^{r} \omega_{\ell i}(f) \cdot s_j^*(s_\ell) = -\sum_{\ell=1}^{r} \omega_{\ell i}(f) \cdot \delta_{j\ell} \\ &= -\omega_{ji}(f).\end{aligned}$$

Thus,

$$\nabla^* s_j^* = -\sum_{k=1}^{r} \omega_{jk}(f) \otimes s_k^*.$$

So, we see that

$$\omega(\nabla^*, f^*) = -\omega(\nabla, f)^T.$$

□

Lemma 1.31. *Let $f = (s_1, \cdots, s_r)^T$ be a frame over U on E and $f^* = (s_1^*, \cdots, s_r^*)^T$ be the dual frame on E^*. Then, $g := f \otimes f^* = (s_i \otimes s_j^*)_{ij}$ is a frame on $E \otimes E^*$.*

Suppose $\xi \in \mathcal{A}^k(X, \mathrm{End}(E))$, then, we have

$$(\nabla^{\mathrm{End}(E)} \xi)(g) = d\xi(g) + \omega(f) \wedge \xi(g) + (-1)^{k+1} \xi(g) \wedge \omega(f).$$

Proof. We first Identify the bundle $\mathrm{End}(E)$ with $E \otimes E^*$. Let ξ be a section of $\bigwedge^k T^*X \otimes \mathrm{End}(E) = \bigwedge^k T^*X \otimes E \otimes E^*$, then we can write

$$\xi = \sum_{j=1}^{r} \sum_{i=1}^{r} \xi_{ij} \otimes (s_i \otimes s_j^*),$$

where $\xi_{ij} \in \mathcal{A}^k(U)$. Let s_{ij} denotes $s_i \otimes s_j^*$.

We now compute

$$\begin{aligned}\nabla^{\mathrm{End}(E)} \xi &= \nabla^{\mathrm{End}(E)} \left(\sum_{i=1}^{r} \sum_{j=1}^{r} \xi_{ij} \otimes (s_i \otimes s_j^*) \right) \\ &= \sum_{i=1}^{r} \sum_{j=1}^{r} \nabla^{\mathrm{End}(E)} (\xi_{ij} \otimes s_{ij}) \\ &= \sum_{i=1}^{r} \sum_{j=1}^{r} \left[d\xi_{ij} \otimes s_{ij} + (-1)^k \xi_{ij} \wedge \nabla^{\mathrm{End}(E)}(s_i \otimes s_j^*) \right] \\ &= \sum_{i=1}^{r} \sum_{j=1}^{r} d\xi_{ij} \otimes s_{ij} + (-1)^k \sum_{i=1}^{r} \sum_{j=1}^{r} \xi_{ij} \wedge \left[(\nabla s_i) \otimes s_j^* + s_i \otimes (\nabla^* s_j^*) \right]\end{aligned}$$

Recall that

$$\nabla s_i = \sum_{\ell=1}^{r} \omega_{\ell i}(f) \otimes s_\ell$$

and

$$\nabla^* s_j^* = -\sum_{k=1}^{r} \omega_{jk}(f) \otimes s_k^*$$

We see that

$$\begin{aligned}(\nabla s_i) \otimes s_j^* + s_i \otimes (\nabla^* s_j^*) &= \left(\sum_{\ell=1}^{r} \omega_{\ell i}(f) \otimes s_\ell\right) \otimes s_j^* - s_i \otimes \left(\sum_{k=1}^{r} \omega_{jk}(f) \otimes s_k^*\right) \\ &= \sum_{\ell=1}^{r} \omega_{\ell i}(f) \otimes (s_\ell \otimes s_j^*) - \sum_{k=1}^{r} \omega_{jk}(f) \otimes (s_i \otimes s_k^*)\end{aligned}$$

So, it follows that

$$\begin{aligned}\nabla^{\mathrm{End}(E)}\xi &= \sum_{i=1}^{r}\sum_{j=1}^{r} d\xi_{ij} \otimes s_{ij} + (-1)^k \sum_{i,j} \xi_{ij} \wedge \left[\sum_{\ell=1}^{r} \omega_{\ell i}(f) \otimes (s_\ell \otimes s_j^*) - \sum_{k=1}^{r} \omega_{jk}(f) \otimes (s_i \otimes s_k^*)\right] \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} d\xi_{ij} \otimes s_{ij} + (-1)^k \sum_{i,j} \left[\sum_{\ell=1}^{r} \xi_{ij} \wedge \omega_{\ell i}(f) \otimes (s_\ell \otimes s_j^*) - \sum_{k=1}^{r} \xi_{ij} \wedge \omega_{jk}(f) \otimes (s_i \otimes s_k^*)\right] \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} d\xi_{ij} \otimes s_{ij} + (-1)^k \sum_{\ell,j} \left(\sum_{i=1}^{r} \xi_{ij} \wedge \omega_{\ell i}(f)\right) \otimes s_{\ell j} - (-1)^k \sum_{i,k} \left(\sum_{j=1}^{r} \xi_{ij} \wedge \omega_{jk}(f)\right) \otimes s_{ik} \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} d\xi_{ij} \otimes s_{ij} + \sum_{\ell,j} \left(\sum_{i=1}^{r} \omega_{\ell i}(f) \wedge \xi_{ij}\right) \otimes s_{\ell j} - (-1)^k \sum_{i,k} \left(\sum_{j=1}^{r} \xi_{ij} \wedge \omega_{jk}(f)\right) \otimes s_{ik}\end{aligned}$$

So, we see that

$$(\nabla^{\mathrm{End}(E)}\xi)(g) = d\xi(g) + \omega(f) \wedge \xi(g) + (-1)^{k+1}\xi(g) \wedge \omega(f).$$

□

Corollary 1.32. *Write $\nabla = d + \omega$ with $\omega \in \mathcal{A}^1(X, \mathrm{End}(E))$, then we have*

$$\nabla^{\mathrm{End}(E)}\xi = d\xi + \omega \wedge \xi + (-1)^{k+1}\xi \wedge \omega$$

for all $\xi \in \mathcal{A}^k(X, \mathrm{End}(E))$.

Proposition 1.33 (Bianchi identity). *Let E be a vector bundle with a connection ∇ and F_∇ be the curvature of ∇, then we have*

$$\nabla^{\mathrm{End}(E)}(F_\nabla) = 0.$$

Proof. Let f, g be the frames in Lemma 1.31.

By (1.3.1), we see that

$$F_\nabla = \sum_{j=1}^r F_\nabla(s_j) \otimes s_j^* = \sum_{j=1}^r \sum_{i=1}^r \Omega_{ij}(f) \otimes s_i \otimes s_j^*.$$

So,

$$F_\nabla(g) = \Omega(f).$$

Thus, we see that

$$\begin{aligned}(\nabla^{\mathrm{End}(E)} F_\nabla)(g) &= dF_\nabla(g) + \omega(f) \wedge F_\nabla(g) + (-1)^{2+1} F_\nabla(g) \wedge \omega(f) \\ &= d\Omega(f) + \omega(f) \wedge \Omega(f) - \Omega(f) \wedge \omega(f) \\ &= \Omega(f) \wedge \omega(f) - \omega(f)\Omega(f) + \omega(f) \wedge \Omega(f) - \Omega(f) \wedge \omega(f) \\ &= 0.\end{aligned}$$

This implies that $\nabla^{\mathrm{End}(E)}(F_\nabla) = 0$. □

Proposition 1.34. *Let E_1 and E_2 be vector bundles endowed with connections ∇_1 and ∇_2, respectively.*

(1) *The curvature of the induced connection on the direct sum $E_1 \oplus E_2$ is given by*

$$F = F_{\nabla_1} \oplus F_{\nabla_2}.$$

(2) *On the tensor product $E_1 \otimes E_2$ the curvature is given by*

$$F_{\nabla_1} \otimes 1 + 1 \otimes F_{\nabla_2}.$$

(3) *For the induced connection ∇^* on the dual bundle E^* one has*

$$F_{\nabla^*} = -F_\nabla^T$$

in the sense

$$\Omega(F_{\nabla^*}, f^*) = -\Omega(\nabla, f)^T,$$

where f^* is the dual frame of f.

(4) *Let $f : M \to N$ be a differentiable map between real manifolds. Let E be a vector bundle on N with a connection ∇. The curvature of the pull-back connection $f^*\nabla$ of ∇ under f is*

$$F_{f^*\nabla} = f^* F_\nabla.$$

Proof. (1) Let ∇ be the connection on $E = E_1 \oplus E_2$ induced by ∇_1 and ∇_2, i.e.

$$\nabla(s_1 \oplus s_2) = \nabla_1(s_1) \oplus \nabla_2(s_2).$$

Then,

$$\begin{aligned}F(s_1 \oplus s_2) &= F_\nabla(s_1 \oplus s_2) \\ &= \nabla(\nabla(s_1 \oplus s_2)) \\ &= \nabla(\nabla_1 s_1 \oplus \nabla_2 s_2) \\ &= \nabla_1^2 s_1 \oplus \nabla_2^2 s_2 \\ &= F_{\nabla_1}(s_1) \oplus F_{\nabla_2}(s_2).\end{aligned}$$

(2) Let ∇ be the connection on $E = E_1 \otimes E_2$ induced by ∇_1 and ∇_2, i.e.

$$\nabla(s_1 \otimes s_2) = \nabla_1(s_1) \otimes s_2 + s_1 \otimes \nabla_2(s_2).$$

Then,

$$\begin{aligned} & F_{\nabla}\left(s_1 \otimes s_2\right) \\ =& \nabla\left(\nabla\left(s_1 \otimes s_2\right)\right) = \nabla\left(\nabla_1\left(s_1\right) \otimes s_2 + s_1 \otimes \nabla_2\left(s_2\right)\right) \\ =& \nabla_1^2\left(s_1\right) \otimes s_2 + (-1)^1 \nabla_1\left(s_1\right) \otimes \nabla_2\left(s_2\right) + \nabla_1\left(s_1\right) \otimes \nabla_2\left(s_2\right) + s_1 \otimes \nabla_2^2\left(s_2\right) \\ =& F_{\nabla_1}\left(s_1\right) \otimes s_2 + s_1 \otimes F_{\nabla_2}\left(s_2\right) \end{aligned}$$

(3) By Lemma 1.30, we see that

$$\nabla^* s_j^* = -\sum_{k=1}^{r} \omega_{jk}(f) \otimes s_k^*.$$

So, we have

$$\begin{aligned} F_{\nabla^*}(s_j^*) &= \nabla^*(\nabla^*(s_j^*)) \\ &= \nabla^*\left(-\sum_{k=1}^{r} \omega_{jk}(f) \otimes s_k^*\right) \\ &= -\sum_{k=1}^{r} \nabla^*\left(\omega_{jk}(f) \otimes s_k^*\right) \\ &= -\sum_{k=1}^{r} \left(d\omega_{jk}(f) \otimes s_k^* - \omega_{jk}(f) \wedge \nabla^* s_k^*\right) \\ &= -\sum_{k=1}^{r} (d\omega_{jk}(f) \otimes s_k^*) + \sum_{k=1}^{r} \left(\omega_{jk}(f) \wedge \nabla^* s_k^*\right) \\ &= -\sum_{k=1}^{r} (d\omega_{jk}(f) \otimes s_k^*) + \omega_{jk}(f) \wedge \sum_{k=1}^{r} \left(-\sum_{\ell=1}^{r} \omega_{k\ell}(f) \otimes s_\ell^*\right) \\ &= -\sum_{\ell=1}^{r} (d\omega_{j\ell}(f) \otimes s_\ell^*) - \sum_{\ell=1}^{r} \left(\sum_{k=1}^{r} \omega_{jk}(f) \wedge \omega_{k\ell}(f)\right) \otimes s_\ell^* \end{aligned}$$

Thus, we see that

$$\Omega(F_{\nabla^*}, f^*) = -d\omega(f)^T - (\omega(f) \wedge \omega(f))^T = -\Omega(\nabla, f)^T.$$

(4) Locally, we have that ∇ is given as $d + \omega$. Then $F_{f^*\nabla} = F_{d + f^*\omega} = d(f^*\omega) + f^*(\omega) \wedge f^*(\omega) = f^*(d\omega + \omega \wedge \omega) = f^* F_{\nabla}$. ☐

1.4 The Chern connection and curvature in holomorphic category

We have already known that given connections ∇_1, ∇_2 on E_1 and E_2 respectively, there exists a natural connection ∇ on the direct sum $E := E_1 \oplus E_2$.

Conversely, let ∇ be a connection on $E = E_1 \oplus E_2$. If we denote by p_1 and p_2 the two projections $E_1 \oplus E_2 \to E_i$. Since every section s_i of E_i can be regarded as a section of E by natural inclusion, we set $\nabla_i(s_i) := (p_i)_*(\nabla(s_i))$. Then, ∇_i is a connection on E_i. Indeed, since $(p_i)_*$ is $\mathcal{A}^0(M)$-linear, we see that

$$\begin{aligned}\nabla_i(f \cdot s_i) &= (p_i)_*(\nabla(f \cdot s_i)) \\ &= (p_i)_*(df \otimes s_i + f \cdot \nabla(s_i)) \\ &= (p_i)_*(df \otimes s_i) + (p_i)_*(f \cdot \nabla(s_i)) \\ &= df \otimes s_i + f \cdot (p_i)_*(\nabla(s_i)) \\ &= df \otimes s_i + f \cdot \nabla_i(s_i).\end{aligned}$$

Thus, we obtain

Lemma 1.35. *The connection ∇ on $E = E_1 \oplus E_2$ induces natural connections ∇_1 and ∇_2 on E_1 and E_2 respectively.*

Let E_1 be a subbundle of E with a given connection ∇ on E.

Definition 1.36. *The* ***second fundamental form*** *of $E_1 \subset E$ with respect to the connection ∇ on E is the section $b \in \mathcal{A}^1(M, \mathrm{Hom}(E_1, E/E_1))$ defined for any local section s of E_1 by*

$$b(s) = (p_{E/E_1})_*(\nabla(s)).$$

The difference of $\nabla_1 \oplus \nabla_2$ and ∇ on $E = E_1 \oplus E_2$ can be measured by the second fundamental form. Indeed, if E splits as $E = E_1 \oplus E_2$ with $E_2 \cong E/E_1$ via the projection, then by definition, $b(s) = (p_2)_*(\nabla(s)) = \nabla(s) - (p_1)_*(\nabla(s)) = \nabla(s) - \nabla_1(s) = \nabla(s) - (\nabla_1 \oplus \nabla_2)(s)$. In this case, $b \in \mathcal{A}^1(M, \mathrm{Hom}(E_1, E_2))$.

Now, consider the decomposition $\mathcal{A}^1(E) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E)$ and a connection ∇ on E, we can decompose ∇ as

$$\nabla = \nabla^{1,0} \oplus \nabla^{0,1},$$

where $\nabla^{1,0} : \mathcal{A}^0(E) \to \mathcal{A}^{1,0}(E)$ and $\nabla^{0,1} : \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$. Note that we have $\nabla^{0,1}(f \cdot s) = \bar{\partial}(f) \otimes s + f \cdot \nabla^{0,1}(s)$. We have the following definition

Definition 1.37. *A connection ∇ on a holomorphic vector bundle E is* ***compatible with the holomorphic structure*** *if $\nabla^{0,1} = \bar{\partial}$.*

Theorem 1.38. *Let (E, h) be a holomorphic vector bundle endowed with a Hermitian structure. Then there exists a unique Hermitian connection ∇ compatible with the holomorphic structure.*

Proof. Let W be a open subset of X and f a holomorphic frame of E. Take a holomorphic section $\xi \in \mathcal{O}_X(W, E)$, we have

$$\begin{aligned}\nabla\xi(f) &= (d + \omega(f))\xi(f) \\ &= (\partial + \omega^{1,0}(f))\xi(f) + (\bar{\partial} + \omega^{0,1}(f))\xi(f),\end{aligned}$$

where $\omega = \omega^{1,0} + \omega^{0,1}$ is the natural decomposition. So,

$$\nabla^{1,0} = (\partial + \omega^{1,0}(f))\xi(f)$$

and

$$\nabla^{0,1} = (\bar{\partial} + \omega^{0,1}(f))\xi(f).$$

Since ξ and f are holomorphic, we see that $\nabla^{0,1}\xi(f) = \omega^{0,1}(f)\xi(f)$. So, we see that ∇ is compatible with the holomorphic structure if and only if the connection matrix ω is of type $(1,0)$.

We first show the uniqueness. Suppose ∇ is a desired connection satisfying the hypothesis. Let $\omega(f)$ be its associated connection matrix with respect to a given frame f over U. Then, by Lemma 1.17, we see that $dh(f) = \omega(f)^T h(f) + h(f)\overline{\omega(f)}$. Since ∇ is compatible with the holomorphic structure, we see that ω is of type $(1,0)$ by the above arguement. So, by comparing the types, we see that $\partial h(f) = \omega(f)^T h(f)$ and $\bar{\partial} h(f) = h(f)\overline{\omega(f)}$. So, this determines $\omega(f) = \overline{h(f)}^{-1}\overline{\partial h(f)}$ uniquely.

We now can construct a Hermitian connection ∇ compatibble with the holomorphic structure by defining the associated connection matrix ω with $\omega(f) := \overline{h(f)}^{-1}\partial\overline{h(f)}$ for a given frame f over U. Then, we see that $\omega(f)^T = (\partial\overline{h(f)})^T(\overline{h(f)}^{-1})^T = (\partial\overline{h(f)})^T h(f)^{-1}$ as $h(f)$ is Hermitian. So, $\omega(f)^T h(f) = (\partial\overline{h(f)})^T = \partial\overline{h(f)}^T = \partial h(f)$, which implies that ω is of $(1,0)$ type and so ∇ is compatible with the holomorphic strucutre. Moreover, we see that $dh(f) = \omega(f)^T h(f) + h(f)\overline{\omega(f)}$. Thus, by Lemma 1.17, we see that the connection ∇ with connection matrix ω is a Hermitian connection.

Now, by the uniqueness, the local pieces glue to a connection globally. $\square$

Definition 1.39. *Let (E,h) be a holomorphic vector bundle endowed with a Hermitian structure. The unique Hermitian connection ∇ compatible with the holomorphic structure is called the **Chern connection** on (E,h).*

Definition 1.40. *Let E be a holomorphic vector bundle on a complex manifold X. A **holomorphic connection** on E is a $\mathbb{C}$-linear map of sheaves*

$$D : E \to \Omega_X \otimes E$$

with

$$D(f \cdot s) = \partial(f) \otimes s + f \cdot D(s)$$

for any local holomorphic function f on X and any local holomorphic section s of E.

Remark. *Here, E denotes both the vector bundle and the sheaf of holomorphic sections of this bundle.*

Now, let E be a holomorphic vector bundle and $X = \bigcup U_i$ be an open covering such that there exists holomorphic trivializations

$$\psi_i : E|_{U_i} \cong U_i \times \mathbb{C}^r.$$

Definition 1.41. *The **Atiyah class***

$$A(E) \in H^1(X, \Omega_X \otimes \mathrm{End}(E))$$

of the holomorphic vector bundle E is given by the Čech cocycle

$$A(E) = \{U_{ij}, \psi_j^{-1} \circ (\psi_{ij}^{-1} d\psi_{ij}) \circ \psi_j\}.$$

Proposition 1.42. *A holomorphic vector bundle E admits a holomorphic connection if and only if its Atiyah class $A(E) \in H^1(X, \Omega_X \otimes \mathrm{End}(E))$ is trivial.*

2 Chern-Weil Theory

2.1 Invariant polynomials

Let V be a complex vector space of dimension n. A k-multilinear symmetric map

$$P : V \times \cdots \times V \to \mathbb{C}$$

corresponds to an element in $\mathrm{Sym}^k(V)^*$. So, one sees that there is a one-to-one correspondence

$$\left\{ \begin{matrix} \text{homogeneous polynomials} \\ \tilde{P} : V \to \mathbb{C} \text{ of degree } k > 1 \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{symmetric } k\text{-multilinear form } P \\ \text{such that } P(X, \cdots, X) = \tilde{P}(X) \end{matrix} \right\},$$

where $X = (x_1, \cdots, x_n)^T \in V$ is a column vector of n variables, via the **polarization identity**

$$P(v_1, \cdots, v_k) = \frac{1}{k!} \sum_{I \subseteq \{1, \cdots, k\}} (-1)^{|I|-k} \tilde{P}\left(\sum_{i \in I} v_i \right).$$

In this section, we will mainly consider the case $V = \mathfrak{gl}(r, \mathbb{C})$, the Lie algebra of complex $r \times r$-matrices.

Definition 2.1. *A symmetric k-multilinear map*

$$P : \mathfrak{gl}(r, \mathbb{C}) \times \cdots \times \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$$

*is called **invariant** if for all $C \in \mathrm{GL}(r, \mathbb{C})$ and all $B_1, \cdots, B_k \in \mathfrak{gl}(r, \mathbb{C})$, we have*

$$P(CB_1C^{-1}, \cdots, CB_kC^{-1}) = P(B_1, \cdots, B_k).$$

Similarly, a polynomial function

$$\tilde{P} : \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$$

*is called **invariant** if for all $C \in \mathrm{GL}(r, \mathbb{C})$ and all $B \in \mathfrak{gl}(r, \mathbb{C})$, we have*

$$\tilde{P}(CBC^{-1}) = \tilde{P}(B).$$

Lemma 2.2. *A symmetric k-multilinear map*

$$P : \mathfrak{gl}(r, \mathbb{C}) \times \cdots \times \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$$

is invariant if and only if its associated homogeneous polynomial

$$\tilde{P} : \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$$

is invariant.

Proof. This follows from the polarization identity. $\square$

Example 2.3. (1) *The determinant function*

$$\det : \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$$

is an invariant polynomial as it is independent of the change of bases.

(2) *The trace function*

$$\mathrm{tr} : \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$$

is an invariant polynomial as it is independent of the change of bases.

Proposition 2.4. *The k-multilinear symmetric map P is invariant if and only if*

$$\sum_{j=1}^{k} P(B_1, \cdots, B_{j-1}, [B, B_j], B_{j+1}, \cdots, B_k) = 0$$

for all $B, B_1, \cdots, B_k \in \mathfrak{gl}(r, \mathbb{C})$.

Proof. $\Rightarrow$: Let $X_j(t) = e^{tB} B_j e^{-tB}$, then

$$\begin{aligned} \frac{d}{dt} P\left(X_1(t), \cdots, X_k(t)\right) &= \sum_{j=1}^{k} P\left(X_1(t), \cdots, \frac{d}{dt} X_j(t), \cdots, X_k(t)\right) \\ &= \sum_{j=1}^{k} P\left(X_1(t), \cdots, B e^{tB} B_j e^{-tB} - e^{tB} B_j B e^{-tB}, \cdots, X_k(t)\right) \end{aligned}$$

Thus,

$$\left.\frac{d}{dt}\right|_{t=0} P\left(X_1(t), \cdots, X_k(t)\right) = \sum_{j=1}^{k} P\left(B_1, \cdots, B B_j - B_j B, \cdots, B_k\right) = 0.$$

$\Leftarrow$: Let $F(t) = P(X_1(t), \cdots, X_k(t))$. If the above equation holds for every B_j and $B \in \mathfrak{gl}(r, \mathbb{C})$, in particular for $X_j(t)$ and B, it follows that

$$F'(t) = 0, \quad \forall t \in \mathbb{R}.$$

Therefore,

$$F(t) = F(0) = P(B_1, \cdots, B_k).$$

This implies that the map $g \mapsto P(gB_1g^{-1}, \cdots, gB_kg^{-1})$ is constant on a neighborhood of $Id \in \mathrm{GL}(r, \mathbb{C})$, for fixed $B_1, \cdots, B_k \in \mathfrak{gl}(r, \mathbb{C})$. But $\mathrm{GL}(r, \mathbb{C})$ is connected Lie group and the considered map is analytic, thus it has to be constant on whole $\mathrm{GL}(r, \mathbb{C})$. ☐

We would like to extend the k-multilinear map to $\mathcal{A}^*(X, \mathrm{End}(E))$, where E is a vector bundle over X.

Let P be an invariant k-multilinear symmetric form on $\mathfrak{gl}(r, \mathbb{C})$. Then for any vector bundle E of rank r and any partition $m = i_1 + \cdots + i_k$, we can define a naturally induced k-linear map

$$P : \mathcal{A}^{i_1}(X, \mathrm{End}(E)) \times \cdots \times \mathcal{A}^{i_k}(X, \mathrm{End}(E)) \to \mathcal{A}^m_{\mathbb{C}}(X)$$

by

$$(\alpha_1 \otimes t_1, \cdots, \alpha_k \otimes t_k) \mapsto (\alpha_1 \wedge \cdots \wedge \alpha_k) P(t_1, \cdots, t_k).$$

Lemma 2.5. *For any forms $\gamma_j \in \mathcal{A}^{i_j}(X, \mathrm{End}(E))$ one has*

$$dP\left(\gamma_1, \cdots, \gamma_k\right) = \sum_{j=1}^{k} (-1)^{\sum_{\ell=1}^{j-1} i_\ell} P\left(\gamma_1, \cdots, \nabla^{\mathrm{End}(E)}\left(\gamma_j\right), \cdots, \gamma_k\right),$$

where $\nabla^{\mathrm{End}(E)}$ denotes the induced connection on $\mathrm{End}(E)$*.*

Proof. We will prove this statement by local calculation. We may write $\nabla = d + \omega$ with $\omega \in \mathcal{A}^1(X, \mathrm{End}(E))$. By Lemma 1.31, we see that

$$\begin{aligned}
dP(\gamma_1, \cdots, \gamma_k) &= \sum_{j=1}^{k} (-1)^{\sum_{\ell=1}^{j-1} i_\ell} P(\gamma_1, \cdots, d\gamma_j, \cdots, \gamma_k) \\
&= \sum_{j=1}^{k} (-1)^{\sum_{\ell=1}^{j-1} i_\ell} P(\gamma_1, \cdots, \nabla^{\mathrm{End}(E)}\gamma_j - \omega \wedge \gamma_j - (-1)^{i_j+1}\gamma_j \wedge \omega, \cdots, \gamma_k) \\
&= \sum_{j=1}^{k} (-1)^{\sum_{\ell=1}^{j-1} i_\ell} P(\gamma_1, \cdots, \nabla^{\mathrm{End}(E)}\gamma_j, \cdots, \gamma_k) - \sum_{j=1}^{k} (-1)^{\sum_{\ell=1}^{j-1} i_\ell} P(\gamma_1, \cdots, \omega \wedge \gamma_j, \cdots, \gamma_k) \\
&\qquad + \sum_{j=1}^{k} (-1)^{\sum_{\ell=1}^{j} i_\ell} P(\gamma_1, \cdots, \gamma_j \wedge \omega, \cdots, \gamma_k)
\end{aligned}$$

It remains to show that

$$-\sum_{j=1}^{k} (-1)^{\sum_{\ell=1}^{j-1} i_\ell} P(\gamma_1, \cdots, \omega \wedge \gamma_j, \cdots, \gamma_k) + \sum_{j=1}^{k} (-1)^{\sum_{\ell=1}^{j} i_\ell} P(\gamma_1, \cdots, \gamma_j \wedge \omega, \cdots, \gamma_k) = 0$$

or equivalently,

$$(-1)^{i_j} \sum_{j=1}^{k} P(\gamma_1, \cdots, \gamma_j \wedge \omega, \cdots, \gamma_k) - \sum_{j=1}^{k} P(\gamma_1, \cdots, \omega \wedge \gamma_j, \cdots, \gamma_k) = 0.$$

We may assume that $\gamma_j = \alpha_j \otimes B_i$ with $\alpha_j \in \mathcal{A}^{i_j}(X)$ and $\omega = \alpha \otimes B$ with $\alpha \in \mathcal{A}^1(X)$. Then,

$$\begin{aligned}
\text{Left Hand Side} &= (-1)^{i_j}(\alpha_1 \wedge \cdots \wedge (\alpha_j \wedge \alpha) \cdots \wedge \alpha_k) \sum_{j=1}^{k} P(B_1, \cdots, [B_j, B], \cdots, B_k) \\
&= 0.
\end{aligned}$$

□

Corollary 2.6. *Let F_∇ be the curvature of an arbitrary connection ∇ on a vector bundle E of rank r. Then for any invariant homogeneous polynomial $\tilde{P}$ of degree k on $\mathfrak{gl}(r, \mathbb{C})$, the induced k-form $\tilde{P}(F_\nabla) \in \mathcal{A}^{2k}_{\mathbb{C}}(X)$ is closed.*

Proof. Let $P : \mathfrak{gl}(r, \mathbb{C}) \times \cdots \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$ be the k-multilinear symmetric map such that

$$P(B, \cdots, B) = \tilde{P}(B).$$

Then,

$$\begin{aligned} d\tilde{P}(F_\nabla) &= dP(F_\nabla, \cdots, F_\nabla) \\ &= \sum_{j=1}^{k} (-1)^{\sum_{\ell=1}^{j-1} i_\ell} P\left(F_\nabla, \cdots, \nabla^{\mathrm{End}(E)}\left(F_\nabla\right), \cdots, F_\nabla\right) \\ &= 0 \end{aligned}$$

as $\nabla^{\mathrm{End}(E)}(F_\nabla) = 0$ by Bianchi identity. □

Thus, for any invariant k-multilinear symmetric map P on $\mathfrak{gl}(r, C)$ and any vector bundle E of rank r, one can associate a de Rham cohomology class $[\tilde{P}(F_\nabla)] \in H^{2k}_{\mathrm{dR}}(X, \mathbb{C})$ as the induced k-form $\tilde{P}(F_\nabla) \in \mathcal{A}^{2k}_{\mathbb{C}}(X)$ is closed. Moreover, this class is independent of the chosen connection due to the following results.

Lemma 2.7. *Let ∇ be a connection on a vector bundle E and $A \in \mathcal{A}^1(X, \mathrm{End}(E))$. Then,*

$$F_{\nabla+A} = F_\nabla + \nabla^{\mathrm{End}(E)}(A) + A \wedge A.$$

Proof. Let ξ be a section on E. Then

$$\begin{aligned} F_{\nabla+A}(\xi) &= (\nabla + A) \circ (\nabla + A)(\xi) \\ &= (\nabla + A)(\nabla\xi + A\xi) \\ &= \nabla^2(\xi) + A(\nabla\xi) + \nabla(A\xi) + (A \wedge A)(\xi) \\ &= F_\nabla(\xi) + A(\nabla\xi) + \nabla(A\xi) + (A \wedge A)(\xi). \end{aligned}$$

It remains to verify that

$$(\nabla^{\mathrm{End}(E)}A)(\xi) = A(\nabla\xi) + \nabla(A\xi).$$

We verify this statement locally. Let $f = (s_1, \cdots, s_r)^T$ be a frame over U on E and $f^* = (s_1^*, \cdots, s_r^*)^T$ be the dual frame on E^*. Then, $g := f \otimes f^* = (s_i \otimes s_j^*)_{ij}$ is a frame on $E \otimes E^*$.

Thus,

$$\begin{aligned} [A(\nabla\xi) + \nabla(A\xi)](f) &= A(g) \wedge [d + \omega(f)]\xi(f) + d[A(g)\xi(f)] + \omega(f) \wedge A(g)\xi(f) \\ &= A(g) \wedge d\xi(f) + A(g) \wedge \omega(f)\xi(f) \\ &\qquad + [dA(g)] \cdot \xi(f) + (-1)^1 A(g) \wedge d\xi(f) + \omega(f) \wedge A(g)\xi(f) \\ &= A(g) \wedge \omega(f)\xi(f) + [dA(g)] \cdot \xi(f) + \omega(f) \wedge A(g)\xi(f) \\ &= (\nabla^{\mathrm{End}(E)}A)(g)\xi(f) \qquad \text{(by Lemma 1.31)} \\ &= [(\nabla^{\mathrm{End}(E)}A)(\xi)](f). \end{aligned}$$

□

Proposition 2.8. *If ∇_0 and ∇_1 are two connections on the same bundle E, then $\tilde{P}(F_{\nabla_0})$ is cohomologous to $\tilde{P}(F_{\nabla_1})$ in de Rham cohomology group, i.e.*

$$[\tilde{P}(F_{\nabla_0})] = [\tilde{P}(F_{\nabla_1})] \in H^{2k}_{\mathrm{dR}}(X, \mathbb{C})$$

if $P : \mathfrak{gl}(r, \mathbb{C}) \times \cdots \times \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$ is a symmetric k-multilinear map.

Proof. Let $\nabla_1 = \nabla_0 + A$ for some $A \in \mathcal{A}^1(X, \mathrm{End}(E))$. Consider a path of connections $\nabla_t := \nabla_0 + tA$ between ∇_0 and ∇_1. Denote by $F_t := F_{\nabla_t}$.

Let us compute the derivative in t at $t = t_0$ of $\tilde{P}(F_t)$. First note that, by Lemma 2.7, we see that

$$F_t = F_{\nabla_t} = F_{\nabla_{t_0}+(t-t_0)A} = F_{t_0} + (t-t_0)\nabla_{t_0}^{\mathrm{End}(E)}A + (t-t_0)^2 A \wedge A,$$

i.e.

$$\frac{F_t - F_{t_0}}{t - t_0} = \nabla_{t_0}^{\mathrm{End}(E)}A + (t-t_0)A \wedge A.$$

$$\left.\frac{d}{dt}\right|_{t=t_0} F_t = \lim_{t\to t_0} \frac{F_t - F_{t_0}}{t-t_0} = \lim_{t\to t_0}\left(\nabla_{t_0}^{\mathrm{End}(E)}A + (t-t_0)A \wedge A\right) = \nabla_{t_0}^{\mathrm{End}(E)}A.$$

Therefore,

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=t_0}\tilde{P}(F_t) &= \left.\frac{d}{dt}\right|_{t=t_0} P(F_t, \cdots, F_t) \\
&= \sum_{j=1}^{k} P\Big(F_{t_0}, \cdots, \left.\frac{d}{dt}\right|_{t=t_0} F_t, \cdots, F_{t_0}\Big) \\
&= \sum_{j=1}^{k} P(F_{t_0}, \cdots, \nabla_{t_0}^{\mathrm{End}(E)}A, \cdots, F_{t_0}) \\
&= kP(F_{t_0}, \cdots, F_{t_0}, \nabla_{t_0}^{\mathrm{End}(E)}A).
\end{aligned}$$

By Lemma 2.5 and Bianchi identity, we see that

$$\begin{aligned}
dP(F_{t_0}, \cdots, F_{t_0}, A) &= \sum_{j=1}^{k-1} P(F_{t_0}, \cdots, \nabla_{t_0}^{\mathrm{End}(E)}(F_{t_0}), \cdots, F_{t_0}, A) + P(F_{t_0}, \cdots, F_{t_0}, \nabla_{t_0}^{\mathrm{End}(E)}(A)) \\
&= P(F_{t_0}, \cdots, F_{t_0}, \nabla_{t_0}^{\mathrm{End}(E)}(A)).
\end{aligned}$$

So, it follows that

$$\left.\frac{d}{dt}\right|_{t=t_0}\tilde{P}(F_t) = dP(F_{t_0}, \cdots, F_{t_0}, kA).$$

Let $\beta_t = P(F_t, \cdots, F_t, kA)$. Then, we see that

$$\left.\frac{d}{dt}\right|_{t=t_0}\tilde{P}(F_t) = d\beta_{t_0}$$

for all $0 \leqslant t_0 \leqslant 1$, i.e.

$$\frac{d}{dt}\tilde{P}(F_t) = d(\beta_t).$$

Thus,

$$\int_0^1 \frac{d}{dt}\tilde{P}(F_t)dt = \int_0^1 d(\beta_t)dt = d\left(\int_0^1 \beta_t dt\right).$$

While,

$$\int_0^1 \frac{d}{dt}\tilde{P}(F_t)dt = \tilde{P}(F_t)\Big|_0^1 = \tilde{P}(F_1) - \tilde{P}(F_0).$$

This implies that $\tilde{P}(F_1) - \tilde{P}(F_0)$ is an exact form. Thus, $[\tilde{P}(F_{\nabla_0})] = [\tilde{P}(F_{\nabla_1})]$. $\square$

To summarize, we have

Theorem 2.9 (Chern-Weil). *Let ∇ be a connection on a vector bundle E of rank r and F_∇ be the curvature of ∇. Suppose*

$$P : \mathfrak{gl}(r, \mathbb{C}) \times \cdots \times \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$$

is an invariant symmetric k-multilinear map. Then,

(1) *The induced k-form $\tilde{P}(F_\nabla) \in \mathcal{A}^{2k}_{\mathbb{C}}(X)$ is closed.*

(2) *The cohomology class $[\tilde{P}(F_\nabla)] \in H^{2k}_{\mathrm{dR}}(X, \mathbb{C})$ is independent of the choice of connection ∇.*

2.2 Chern classes, Chern characters and their properties

As we know, the determinant function $\det : \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$ is an invariant polynomial. So, the function

$$B \mapsto \det(I + B)$$

is also an invariant polynomial. Let $\{\tilde{P}_k\}$ be the homogeneous components of $B \mapsto \det(I + B)$, i.e. $\{\tilde{P}_k\}$ are homogenous polynomials defined by

$$\det(I + B) = 1 + \tilde{P}_1(B) + \cdots + \tilde{P}_r(B).$$

Then, $\tilde{P}_k(B)$ are also invariant polynomials.

Now, let E be a vector bundle of rank r with a connection ∇ over a real manifold X. Let F_∇ be the curvature of ∇.

Definition 2.10. *The closed differential form*

$$c_k(E, \nabla) := \tilde{P}_k\left(\frac{i}{2\pi} F_\nabla\right) \in \mathcal{A}^{2k}_{\mathbb{C}}(X)$$

*is called the **k-th Chern form** of (E, ∇).*

Definition 2.11. *The **k-th Chern class** of E is defined to be the induced cohomology class*

$$c_k(E) := [c_k(E, \nabla)] \in H^{2k}_{\mathrm{dR}}(X, \mathbb{C}).$$

In particular, $c_0(E) = 1$ and $c_k(E) = 0$ for $k > r$.

*The **total Chern class** of E is*

$$c(E) := c_0(E) + \cdots + c_r(E) \in H^{2*}_{\mathrm{dR}}(X, \mathbb{C}).$$

Similarly, the trace function $\mathrm{tr} : \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$ is an invariant polynomial, which induces an invariant map

$$B \mapsto \mathrm{tr}(e^B).$$

Let $\{\tilde{Q}_k\}$ be the homogeneous polynomials of degree k defined by

$$\operatorname{tr}(e^B) = \tilde{Q}_0(B) + \tilde{Q}_1(B) + \cdots + \tilde{Q}_k(B) + \cdots$$

Definition 2.12. *The* ***k-th Chern character*** $\operatorname{ch}_k(E) \in H^{2k}_{\mathrm{dR}}(X, \mathbb{C})$ *of E is defined as the cohomology class*

$$\operatorname{ch}_k(E) = [\operatorname{ch}_k(E, \nabla)],$$

where

$$\operatorname{ch}_k(E, \nabla) := \tilde{Q}_k\left(\frac{i}{2\pi} F_\nabla\right) \in \mathcal{A}^{2k}_{\mathbb{C}}(X).$$

The ***total Chern character*** *is*

$$\operatorname{ch}(E) := \operatorname{ch}_0(E) + \cdots + \operatorname{ch}_r(E) + \operatorname{ch}_{r+1}(E) + \cdots$$

Now, if we consider another function

$$B \mapsto \frac{\det(tB)}{\det(I - e^{-tB})},$$

we obtain a collection of polynomials $\{\tilde{T}_k\}$ defined by the expansion

$$\frac{\det(tB)}{\det(I - e^{-tB})} = \sum_k \tilde{T}_k(B) t^k.$$

Clearly, $\tilde{T}_k$ is homogeneous of degree k and invariant.

Definition 2.13. *The* ***k-th Todd class*** $\operatorname{td}_k(E) \in H^{2k}_{\mathrm{dR}}(X, \mathbb{C})$ *of E is defined as the cohomology class*

$$\operatorname{td}_k(E) = [\operatorname{td}_k(E, \nabla)],$$

where

$$\operatorname{td}_k(E, \nabla) := \tilde{T}_k\left(\frac{i}{2\pi} F_\nabla\right) \in \mathcal{A}^{2k}_{\mathbb{C}}(X).$$

The ***total Todd class*** *is*

$$\operatorname{td}(E) := \operatorname{td}_0(E) + \cdots + \operatorname{td}_r(E) + \operatorname{td}_{r+1}(E) + \cdots$$

Let us now study some of the natural operations for vector bundles and see how the characteristic classes behave in these situations.

Proposition 2.14. *Let $E = E_1 \oplus E_2$ be endowed with the direct sum ∇ of the connections ∇_1 and ∇_2 on E_1 and E_2 respectively. Then,*

(1) $c(E, \nabla) = c(E_1, \nabla_1) \cdot c(E_2, \nabla_2)$.
(2) $c(E) = c(E_1) \cdot c(E_2)$.
(3) $\operatorname{ch}(E) = \operatorname{ch}(E_1) + \operatorname{ch}(E_2)$.

Proof. (1) The curvature F_∇ of ∇ satisfies $F_\nabla = F_{\nabla_1} \oplus F_{\nabla_2}$. Thus,

$$\begin{aligned} c(E,\nabla) &= \det\left(I_E + \frac{i}{2\pi}F_\nabla\right) \\ &= \det\left((I_{E_1} + \frac{i}{2\pi}F_{\nabla_1}) \oplus (I_{E_2} + \frac{i}{2\pi}F_{\nabla_2})\right) \\ &= \det\left(I_{E_1} + \frac{i}{2\pi}F_{\nabla_1}\right) \cdot \det\left(I_{E_2} + \frac{i}{2\pi}F_{\nabla_2}\right) \\ &= c(E_1,\nabla_1)\cdot c(E_2,\nabla_2). \end{aligned}$$

(2) $c(E) = [c(E,\nabla)] = [c(E_1,\nabla_2)\cdot c(E_2,\nabla_2)] = [c(E_1,\nabla_1)]\cdot[c(E_2,\nabla_2)] = c(E_1)\cdot c(E_2)$.
(3) Since

$$\begin{aligned} \mathrm{ch}(E,\nabla) &= \mathrm{tr}\left(e^{\frac{i}{2\pi}F_\nabla}\right) \\ &= \mathrm{tr}\left(e^{\frac{i}{2\pi}F_{\nabla_1}} \oplus e^{\frac{i}{2\pi}F_{\nabla_2}}\right) \\ &= \mathrm{tr}\left(e^{\frac{i}{2\pi}F_{\nabla_1}}\right) + \mathrm{tr}\left(e^{\frac{i}{2\pi}F_{\nabla_2}}\right) \\ &= \mathrm{ch}(E_1,\nabla_1) + \mathrm{ch}(E_2,\nabla_2). \end{aligned}$$

Thus, it follows that $\mathrm{ch}(E) = \mathrm{ch}(E_1) + \mathrm{ch}(E_2)$. □

Corollary 2.15. (1) $c_k(E_1 \oplus E_2) = \sum_{i=0}^{k} c_i(E_1) \cup c_{k-i}(E_2)$.
(2) $\mathrm{ch}_k(E_1 \oplus E_2) = \mathrm{ch}_k(E_1) + \mathrm{ch}_k(E_2)$.

Proof. Simply by comparing the degree. □

Proposition 2.16. *Let E_1 and E_2 be two vector bundles, then*

$$\mathrm{ch}(E_1 \otimes E_2) = \mathrm{ch}(E_1)\cdot \mathrm{ch}(E_2).$$

Proof. Let ∇_1 and ∇_2 be connections on E_1 and E_2 respectively. Let ∇ be the induced connection $\nabla_1 \otimes 1 + 1 \otimes \nabla_2$ on the tensor product $E = E_1 \otimes E_2$. By Proposition 1.34(2), we see that $F_\nabla = F_{\nabla_1} \otimes 1 + 1 \otimes F_{\nabla_2}$. Recall that $e^A = \sum_{k=0}^{\infty} \frac{1}{k!}A^k$, we have

$$\begin{aligned} e^{\lambda(F_{\nabla_1}\otimes 1 + 1\otimes F_{\nabla_2})}(s_1 \otimes s_2) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\left(F_{\nabla_1}\otimes 1 + 1 \otimes F_{\nabla_2}\right)^k (s_1 \otimes s_2) \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\left(\sum_{i=0}^{k}\binom{k}{i} F_{\nabla_1}^i(s_1) \otimes F_{\nabla_2}^{k-i}(s_2)\right) \\ &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} \frac{\lambda^i}{i!}F_{\nabla_1}^i(s_1) \otimes \frac{\lambda^{k-i}}{(k-i)!}F_{\nabla_2}^{k-i}(s_2)\right) \\ &= \left[\sum_{i=0}^{\infty}\frac{\lambda^i}{i!}F_{\nabla_1}^i(s_1)\right] \otimes \left[\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}F_{\nabla_2}^j(s_2)\right] \\ &= e^{\lambda F_{\nabla_1}}(s_1) \otimes e^{\lambda F_{\nabla_2}}(s_2) \\ &= e^{\lambda F_{\nabla_1}} \otimes e^{\lambda F_{\nabla_2}}(s_1 \otimes s_2) \end{aligned}$$

for all sections s_1, s_2 on E_1 and E_2 respectively. Thus,

$$e^{\lambda(F_{\nabla_1}\otimes 1 + 1\otimes F_{\nabla_2})} = e^{\lambda F_{\nabla_1}} \otimes e^{\lambda F_{\nabla_2}}.$$

Thus, we see that

$$\begin{aligned}
\mathrm{ch}(E) &= \mathrm{tr}\left(e^{\frac{i}{2\pi}F_{\nabla}}\right) = \mathrm{tr}\left(e^{\frac{i}{2\pi}(F_{\nabla_1}\otimes 1 + 1\otimes F_{\nabla_2})}\right) \\
&= \mathrm{tr}\left(e^{\frac{i}{2\pi}F_{\nabla_1}} \otimes e^{\frac{i}{2\pi}F_{\nabla_2}}\right) \\
&= \mathrm{tr}\left(e^{\frac{i}{2\pi}F_{\nabla_1}}\right) \cdot \mathrm{tr}\left(e^{\frac{i}{2\pi}F_{\nabla_2}}\right) \\
&= \mathrm{ch}(E_1) \cdot \mathrm{ch}(E_2).
\end{aligned}$$

So, we conclude that

$$\mathrm{ch}(E_1 \otimes E_2) = \mathrm{ch}(E_1) \cdot \mathrm{ch}(E_2).$$

□

Proposition 2.17. *Let E be a vector bundle of rank r and L a line bundle, then*

$$c_i(E \otimes L) = \sum_{j=0}^{i} \binom{r-j}{i-j} c_j(E) \cup c_1(L)^{i-j}.$$

Proof. content... □

Proposition 2.18. *Let E be a vector bundle with a connection ∇ and E^* be the dual bundle with the natural connection ∇^*. Then,*

$$c_k(E^*, \nabla^*) = (-1)^k c_k(E, \nabla).$$

In particular,

$$c_k(E^*) = (-1)^k c_k(E).$$

Proof. Recall that by Proposition 1.34(3), we have $F_{\nabla^*} = -F_{\nabla}^T$. Thus, we see that

$$\det\left(I + \frac{i}{2\pi}F_{\nabla^*}\right) = \det\left(I - \frac{i}{2\pi}F_{\nabla}^T\right) = \det\left(I - \frac{i}{2\pi}F_{\nabla}\right)$$

Let $\{\tilde{P}_k\}$ be the homogeneous components of $B \mapsto \det(I + B)$, i.e. $\{\tilde{P}_k\}$ are homogenous polynomials defined by

$$\det(I + B) = 1 + \tilde{P}_1(B) + \cdots + \tilde{P}_r(B).$$

Then, we have

$$c_k(E^*, \nabla^*) = \tilde{P}_k\left(\frac{i}{2\pi}F_{\nabla^*}\right) = \tilde{P}_k\left(-\frac{i}{2\pi}F_{\nabla}\right) = (-1)^k \tilde{P}_k\left(\frac{i}{2\pi}F_{\nabla}\right) = (-1)^k c_k(E, \nabla).$$

Take cohomology class, we obtain

$$c_k(E^*) = (-1)^k c_k(E).$$

□

Proposition 2.19. *Let $f : M \to N$ be a differentiable map between real manifolds and let E be a vector bundle on N endowed with a connection ∇. Then,*

$$c_k(f^*E, f^*\nabla) = f^*c_k(E, \nabla).$$

Proof. By Proposition 1.34(4), we have $F_{f^*\nabla} = f^*F_\nabla$. Let $\{\tilde{P}_k\}$ be the homogeneous components of $B \mapsto \det(I + B)$, i.e. $\{\tilde{P}_k\}$ are homogenous polynomials defined by

$$\det(I + B) = 1 + \tilde{P}_1(B) + \cdots + \tilde{P}_r(B).$$

Then, we have

$$c_k(f^*E, f^*\nabla) = \tilde{P}_k\left(\frac{i}{2\pi}F_{f^*\nabla}\right) = \tilde{P}_k\left(\frac{i}{2\pi}f^*F_\nabla\right) = f^*\tilde{P}_k\left(\frac{i}{2\pi}F_\nabla\right) = f^*c_k(E, \nabla).$$

□

Proposition 2.20. *The first Chern class of the line bundle $\mathcal{O}(1)$ on $\mathbb{CP}^1$ satisfies the normalization*

$$\int_{\mathbb{CP}^1} c_1(\mathcal{O}(1)) = 1.$$

Proof. content... □

Proposition 2.21. *Let E be a vector bundle, then the total Chern class is real, i.e.*

$$c(E) \in H^*(X, \mathbb{R}).$$

Proof. Pick an Hermitian metric on the vector bundle E and consider an Hermitian connection ∇, which always exists. Then locally and with respect to an Hermitian trivialization of E the curvature satisfies the equation

$$F_\nabla^* = \overline{F_\nabla}^T = -F_\nabla.$$

Thus, we see that

$$\overline{\frac{i}{2\pi}F_\nabla} = \frac{i}{2\pi}F_\nabla^T.$$

So,

$$\begin{aligned} c(E, \nabla) &= \det\left(I + \frac{i}{2\pi}F_\nabla\right) \\ &= \det\left(I + \frac{i}{2\pi}F_\nabla^T\right) \\ &= \det\left(I + \overline{\frac{i}{2\pi}F_\nabla}\right) \\ &= \overline{\det\left(I + \frac{i}{2\pi}F_\nabla\right)} \\ &= \overline{c(E, \nabla)}. \end{aligned}$$

We see that $c(E, \nabla)$ is a real form. Thus,

$$c(E) \in H^*(X, \mathbb{R}).$$

□

Definition 2.22. *Let E be a vector bundle of rank r over X. A **splitting map** $f : Y \to X$ for E is a map such that*

$$f^*E = L_1 \oplus \cdots \oplus L_n$$

*is the whitney sum of line bundles L_i and $f^* : H^*(X) \to H^*(Y)$ is an injective map. We call Y a **splitting manifold** of E.*

Proposition 2.23 (Splitting principle). *Every vector bundle E of finite rank over X admits a splitting map $f : Y \to X$ with Y a splitting manifold of E.*

We will not prove this result at this moment.

2.3 Comparison of approaches to the first Chern class