Čech Cohomology and Sheaf Cohomology

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June 24, 2020

Contents

1 Review of sheaf theory and homological algebra

Definition 1.1. *Let X be a topological space. A* ***presheaf $\mathscr{F}$ of abelian groups*** *on X consists of the data*

(*a*) *for every open subset $U \subseteq X$, an abelian group $\mathscr{F}(U)$, and*

(*b*) *for every inclusion $V \subseteq U$ of open subsets of X, a homomorphism of abelian groups $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$,*

subject to the conditions

(0) *$\mathscr{F}(\varnothing) = 0$, where $\varnothing$ is the empty set,*

(1) *ρ_{UU} is the identity map $\mathscr{F}(U) \to \mathscr{F}(U)$, and*

(2) *if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.*

Remark. For any topological space X, we define a category $\mathfrak{Top}(X)$, whose objects are the open subsets of X, and where the only morphisms are the inclusion maps. Thus $\mathrm{Hom}(V, U)$ *is empty* *if $V \nsubseteq U$, and $\operatorname{Hom}(V,U)$ has just one element if $V \subseteq U$. Now a presheaf is just a contravariant functor from the category $\mathfrak{Top}(X)$ to the category $\mathfrak{Ab}$ of abelian groups.*

We define a presheaf of rings, a presheaf of sets, or a presheaf with values in any fixed category $\mathfrak{C}$, by replacing the words "abelian group" in the definition by "ring", "set", or "object of $\mathfrak{C}$" respectively.

Definition 1.2. *If $\mathscr{F}$ is a presheaf on X, we refer to $\mathscr{F}(U)$ as the* ***sections*** *of the presheaf $\mathscr{F}$ over the open set U, and we sometimes use the notation $\Gamma(U,\mathscr{F})$ to denote the group $\mathscr{F}(U)$. We call the maps ρ_{UV} restriction maps, and we sometimes write $s|_V$ instead of $\rho_{UV}(s)$, if $s \in \mathscr{F}(U)$.*

Definition 1.3. *A presheaf $\mathscr{F}$ on a topological space X is a* ***sheaf*** *if it satisfies the following supplementary conditions:*

(3) *if U is an open set, if $\{V_i\}$ is an open covering of U, and if $s \in \mathscr{F}(U)$ is an element such that $s|_{V_i} = 0$ for all i, then $s = 0$;*

(4) *if U is an open set, if $\{V_i\}$ is an open covering of U, and if we have elements $s_i \in \mathscr{F}(V_i)$ for each i, with the property that for each i,j, $s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}$ then there is an element $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$ for each i. (Note condition (3) implies that s is unique.)*

Definition 1.4. *If $\mathscr{F}$ is a presheaf on X, and if P is a point of X, we define the* ***stalk*** *$\mathscr{F}_P$ of $\mathscr{F}$ at P to be the direct limit of the groups $\mathscr{F}(U)$ for all open sets U containing P, via the restriction maps ρ. Here $U \leqslant V \Leftrightarrow V \subseteq U$.*

Remark. *By the definition of direct limit, an element of $\mathscr{F}_P$ is represented by a pair $\langle U, s\rangle$, where U is an open neighborhood of P, and s is an element of $\mathscr{F}(U)$. Two such pairs $\langle U, s\rangle$ and $\langle V, t\rangle$ define the same element of $\mathscr{F}_P$ if and only if there is an open neighborhood W of P with $W \subseteq U \cap V$, such that $s|_W = t|_W$. Thus we may speak of elements of the stalk $\mathscr{F}_P$ as germs of sections of $\mathscr{F}$ at the point P.*

Definition 1.5. *If $\mathscr{F}$ and $\mathscr{G}$ are presheaves on X, a* ***morphism*** *$\varphi : \mathscr{F} \to \mathscr{G}$ consists of a morphism of abelian groups $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ for each open set U, such that whenever $V \subseteq U$ is an inclusion, the diagram*

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_{UV}"'] & \mathscr{G}(U) \arrow[d, "\rho'_{UV}"] \\
\mathscr{F}(V) \arrow[r, "\varphi(V)"'] & \mathscr{G}(V)
\end{tikzcd}$$

is commutative, where ρ and ρ' are the restriction maps in $\mathscr{F}$ and $\mathscr{G}$. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves on X, we use the same definition for a morphism of sheaves. An ***isomorphism*** *is a morphism which has a two-sided inverse.*

Remark. *A morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of presheaves on X induces a morphism $\varphi_P : \mathscr{F}_P \to \mathscr{G}_P$ on the stalks, for any point $P \in X$.*

Proposition 1.6. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on a topological space X. Then φ is an isomorphism if and only if the induced map on the stalk $\varphi_P : \mathscr{F}_P \to \mathscr{G}_P$ is an isomorphism for every $P \in X$.*

Definition 1.7. *Given a presheaf $\mathscr{F}$, there is a sheaf $\mathscr{F}^+$ and a morphism $\theta : \mathscr{F} \to \mathscr{F}^+$, with the property that for any sheaf $\mathscr{G}$, and any morphism $\varphi : \mathscr{F} \to \mathscr{G}$, there is a unique morphism $\psi : \mathscr{F}^+ \to \mathscr{G}$ such that the following diagram commutes*

$$\begin{tikzcd} & \mathscr{G} & \\ \mathscr{F} \arrow[ur, "\varphi"] \arrow[rr, "\theta"'] & & \mathscr{F}^+ \arrow[ul, dotted, "\psi"'] \end{tikzcd}$$

$\varphi = \psi \circ \theta$. *Furthermore the pair $(\mathscr{F}^+, \theta)$ is unique up to unique isomorphism. $\mathscr{F}^+$ is called* ***the sheaf associated to the presheaf $\mathscr{F}$.***

Definition 1.8. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. We define the* ***presheaf kernel*** *of φ,* ***presheaf cokernel*** *of φ, and* ***presheaf image*** *of φ to be the presheaves given by $U \mapsto \ker(\varphi(U))$, $U \mapsto \operatorname{coker}\,(\varphi(U))$, and $U \mapsto \operatorname{Im}\,\varphi(U)$ respectively.*

Remark. *If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then the presheaf kernel of φ is a sheaf, but the presheaf cokernel and presheaf image of φ are in general not sheaves.*

Definition 1.9. *If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, we define the* ***kernel*** *of φ, denoted $\ker\varphi$, to be the presheaf kernel of φ, which is a sheaf. Thus $\ker\varphi$ is a subsheaf of $\mathscr{F}$.*

We say that a morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ is ***injective*** *if $\ker\varphi = 0$. Thus φ is injective if and only if the induced map $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for every open set of X.*

Definition 1.10. *If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, we define the* ***image*** *of φ, denoted $\operatorname{im}\,\varphi$, to be the sheaf associated to the presheaf image of φ.*

We say that a morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of sheaves is ***surjective*** *if $\operatorname{im}\,\varphi = \mathscr{G}$.*

Definition 1.11. *We say that a sequence $\cdots \longrightarrow \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \longrightarrow \cdots$ of sheaves and morphisms is* ***exact*** *if at each stage $\ker\varphi^i = \operatorname{im}\,\varphi^{i-1}$.*

Remark. $0 \longrightarrow \mathscr{F} \xrightarrow{\varphi} \mathscr{G}$ *is exact if and only if φ is injective, and $\mathscr{F} \xrightarrow{\varphi} \mathscr{G} \longrightarrow 0$ is exact if and only if φ is surjective.*

Proposition 1.12. *Let $\{\mathcal{F}_i\}_{i\in I}$ be a family of a sheaf. Define a presheaf $\prod\limits_{i\in I} \mathcal{F}_i$ by*

$$U \mapsto \prod_{i\in I} \mathcal{F}_i(U).$$

Then the presheaf $\prod\limits_{i\in I} \mathcal{F}_i$ is a sheaf.

Proof. Let $\{V_j\}_j$ be an open cover of U.

(1) If $(s_i)_i \in \prod_{i\in I} \mathcal{F}_i(U)$ such that $(s_i)|_{V_j} = 0$ for all j. Then for any fixed $i \in I$, we have $s_i|_{V_j} = 0$ for all j. Thus, $s_i = 0 \in \mathcal{F}_i(U)$ as $\mathcal{F}_i$ is a sheaf. So, $(s_i)_i = 0$.

(2) Let $s_j = (s_{ij})_{i\in I} \prod_{i\in I} \mathcal{F}_i(V_j)$ such that $s_j|_{V_j\cap V_k} = s_k|_{V_j\cap V_k}$ for all j, k. That is, $(s_{ij})|_{V_j\cap V_k} = (s_{ik})|_{V_j\cap V_k}$. So, for any fixed $i \in I$, we have $s_{ij}|_{V_j\cap V_k} = s_{ik}|_{V_j\cap V_k}$. Since $\mathcal{F}_i$ is a sheaf, there exists some $s_i \in \mathcal{F}_i(U)$ such that $s_i|_{V_j} = s_{ij}$ for all j. Take $s = (s_i)_i \in \prod_{i\in I} \mathcal{F}_i(U)$, we see that $s|_{V_j} = (s_i)|_{V_j} = (s_i|_{V_j}) = (s_{ij}) = s_j$ for each j. $\square$

2 Cohomology of sheaves

2.1 Injective sheaves and flasque sheaves

Definition 2.1. *A sheaf $\mathcal{I}$ is* ***injective*** *if for any injective sheaf map $h : \mathcal{F} \to \mathcal{G}$ and any sheaf map $f : \mathcal{F} \to \mathcal{I}$, there is some sheaf map $\hat{f} : \mathcal{G} \to \mathcal{I}$ extending $f : \mathcal{F} \to \mathcal{I}$ in the sense that $f = \hat{f} \circ h$, as in the following commutative diagram:*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r, "h"] \arrow[d, "f"'] & \mathcal{G} \arrow[dl, dotted, "\hat{f}"] \\ & \mathcal{I} & \end{tikzcd}$$

Equivalently, this means that the contravariant functor $\mathrm{Hom}_{\mathbf{Sh}(X)}(-, \mathcal{I})$ *is exact.*

We know that the category of R-modules has enough injectives. This will imply that the category of sheaves of R-modules also has enough injectives.

Proposition 2.2. *For any sheaf $\mathcal{F}$ of R-modules, there is an injective sheaf $\mathcal{I}$ and an injective sheaf homomorphism $\varphi : \mathcal{F} \to \mathcal{I}$.*

Proof. For every $x \in X$, pick some injection $\mathcal{F}_x \to I^x$ with I^x an injective R-module, which always exists. Define the "skyscraper sheaf" $\mathcal{I}^x$ as the sheaf given by

$$\mathcal{I}^x(U) = \begin{cases} I^x, & \text{if } x \in U, \\ 0, & \text{if } x \notin U \end{cases}$$

for every open subset $U \subseteq X$. It is easy to check that there is an isomorphism

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(\mathcal{F}, \mathcal{I}^x) \cong \mathrm{Hom}_R(\mathcal{F}_x, I^x)$$

for any sheaf $\mathcal{F}$, and this implies that $\mathcal{I}^x$ is an injective sheaf. We also have a sheaf map from $\mathcal{F}$ to $\mathcal{I}^x$. Consequently we obtain an injective sheaf map

$$\mathcal{F} \to \prod_{x \in X} \mathcal{I}^x.$$

Since a product of injective sheaves is injective, $\mathcal{F}$ is embedded into an injective sheave. □

Remark. The category of sheaves does have enough projectives. This is the reason why projective resolutions of sheaves are of little interest.

Definition 2.3. *Let X be a topological space, and let $\Gamma(X, -)$ be the global section functor from the abelian category $\mathbf{Sh}(X)$ of sheaves of R-modules to the category of abelian groups. The* ***cohomology groups*** *of the sheaf $\mathcal{F}$ (or the* ***cohomology groups*** *of X with values in $\mathcal{F}$), denoted by $\mathrm{H}^p(X, \mathcal{F})$, are the groups $R^p\Gamma(X, -)(\mathcal{F})$ induced by the right derived functor $R^p\Gamma(X, -)$ (with $p \geqslant 0$).*

To compute the sheaf cohomology groups $\mathrm{H}^p(X, \mathcal{F})$, pick any resolution of $\mathcal{F}$

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \xrightarrow{d^0} \mathcal{I}^1 \xrightarrow{d^1} \mathcal{I}^2 \xrightarrow{d^2} \cdots$$

by injective sheaves $\mathcal{I}^n$. Apply the global section functor $\Gamma(X, -)$ to obtain the complex of R-modules

$$0 \xrightarrow{\delta^{-1}} \mathcal{I}^0(X) \xrightarrow{\delta^0} \mathcal{I}^1(X) \xrightarrow{\delta^1} \mathcal{I}^2(X) \xrightarrow{\delta^2} \cdots$$

and then

$$\mathrm{H}^p(X, \mathcal{F}) = \ker \delta^p / \mathrm{Im}\delta^{p-1}.$$

We now turn to flasque sheaves.

Definition 2.4. *Let X be a topological space. A sheaf $\mathcal{F}$ on X is* ***flasque*** *if for every open subseteq $V \subseteq U$, the restriction map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.*

Proposition 2.5. *A sheaf $\mathcal{F}$ is flasque if and only if for every open subset U of X, the restriction map $\rho_{XU} : \mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.*

Proof. $\Rightarrow$: By definition.

$\Leftarrow$: Let $V \subseteq U$ be open subsets of X. Then consider the following diagram

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r, "\rho_{XU}"] \arrow[dr, "\rho_{XV}"'] & \mathcal{F}(U) \arrow[d, "\rho_{UV}"] \\ & \mathcal{F}(V). \end{tikzcd}$$

We see that ρ_{UV} is surjective as ρ_{XU} and ρ_{XV} are. □

Proposition 2.6. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. If $\mathcal{F}$ is flasque, so is $\mathcal{F}|_U$ for every open subset U of X. Conversely, if for every $x \in X$, there is a neighborhood U such that $\mathcal{F}|_U$ is flasque, then $\mathcal{F}$ is flasque.*

Proof. $\Rightarrow$: By definiton.

$\Leftarrow$: Given any open subset V of X, take $s \in \mathcal{F}(V)$. Let

$$T = \{(U, t) : U \text{ is open in } X \text{ such that } V \subseteq U \text{ and } t \in \mathcal{F}(U) \text{ such that } t|_V = s\}.$$

We define a partial order $\leqslant$ on T by

$$(U_1, t_1) \leqslant (U_2, t_2) \Leftrightarrow U_1 \subseteq U_2 \text{ and } t_2|_{U_1} = t_1.$$

Let (U_i, t_i) be a chain in T. Let $U = \bigcup_i U_i$, then there exists $t \in \mathcal{F}(U)$ such that $t|_{U_i} = t_i$ by the gluability of sheaves. We see that (U, t) is a upper bound of (U_i, t_i). By Zorn's lemma, there exists a maximal element (U_0, t_0) in T. If $U_0 \neq X$, there exists a point $x \in X - U_0$. Then there exists a neighborhood W of x such that $\mathcal{F}|_W$ is flasque. We see that $W \not\subseteq U_0$. We now can extend the section $\rho_{U_0, U_0 \cap W}(t_0)$ to $t' \in \mathcal{F}(W)$ as $\mathcal{F}(W) \to \mathcal{F}(U_0 \cap W)$ is surjective. Since t_0 and t' agree on $U_0 \cap W$, we can glue them to obtain a section t on $U_0 \cup W$. Then $(U_0, t_0) \leqslant (U_0 \cup W, t)$ and $(U_0 \cup W, t) \in T$. Contradiction. This imples that $U_0 = X$. So, we see that $\mathcal{F}(X) \to \mathcal{F}(V)$ is surjective. By Proposition 2.5, we see that $\mathcal{F}$ is flasque. □

Lemma 2.7. *If $(X, \mathcal{O}_X)$ is a ringed space, any injective $\mathcal{O}_X$-module is flasque.*

Proof. For any open subset $U \subseteq X$, we define the sheaf $\mathcal{O}_U$ by

$$\mathcal{O}_U(V) = \begin{cases} \mathcal{O}_X|_U(V), & \text{if } V \subseteq U, \\ 0, & \text{otherwise.} \end{cases}$$

We see that

$$\mathcal{O}_{U,p} = \begin{cases} \mathcal{O}_{X,p}, & \text{if } p \in U, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $\mathcal{I}$ is an injective $\mathcal{O}_X$-module and $V \subseteq U$ are open subsets. Then, we have an injective inclusion

$$0 \to \mathcal{O}_V \to \mathcal{O}_U.$$

Since $\mathcal{I}$ is an injective sheaf, the functor $\mathrm{Hom}_{\mathbf{Sh}(X)}(-, \mathcal{I})$ is exact. Thus,

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{I}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{I}) \to 0$$

is exact.

Since $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{I}) \cong \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{O}_X|_U, \mathcal{I}|_U) \cong \mathcal{I}(U)$, we see that $\mathcal{I}(U) \to \mathcal{I}(V) \to 0$ is exact. Thus, $\mathcal{I}$ is flasque. $\square$

So far, we see that every $\mathcal{O}_X$-module $\mathcal{F}$ admits a flasque resolution. Further, there is a canonical way to construct a flasque resolution of $\mathcal{F}$, called **canonical flasque resolution** or **Godement resolution** of $\mathcal{F}$.

Define a presheaf $C^0(X, \mathcal{F})$ by

$$U \mapsto \prod_{x \in U} \mathcal{F}_x.$$

(To be continued...)

Given two sheaves of R-modules $\mathcal{F}'$ and $\mathcal{F}''$, we obtain a presheaf $\mathcal{F}' \oplus \mathcal{F}''$ by setting

$$\mathcal{F}(U) = (\mathcal{F}' \oplus \mathcal{F}'')(U) = \mathcal{F}'(U) \oplus \mathcal{F}''(U)$$

for every open subset U of X. Actually, $\mathcal{F}' \oplus \mathcal{F}''$ is a sheaf. We call $\mathcal{F}'$ and $\mathcal{F}''$ **direct summands** of $\mathcal{F}$.

Proposition 2.8. *Let* $0 \longrightarrow \mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \longrightarrow 0$ *be an exact sequence of sheaves and* $\mathcal{F}'$ *be flasque. Then for every open subset* $U \subseteq X$*, we have an exact sequence*

$$0 \longrightarrow \mathcal{F}'(U) \xrightarrow{\varphi(U)} \mathcal{F}(U) \xrightarrow{\psi(U)} \mathcal{F}''(U) \longrightarrow 0.$$

Equivalently,

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \longrightarrow 0$$

is an exact sequence of presheaves.

Proof. It suffices to show that $\psi(U) : \mathcal{F}(U) \to \mathcal{F}''(U)$ is surjective. Let $t \in \mathcal{F}''(U)$. Recall that for any $x \in U$, we have $\psi_x : \mathcal{F}_x \to \mathcal{F}''_x$ is surjective, i.e. there exists some $s_x \in \mathcal{F}_x$ such that $\psi_x(s_x) = t_x$. Thus, there exists a neighborhood U_x of x and $s_{U_x} \in \mathcal{F}(U_x)$ such that $\psi(U_x)(s_{U_x}) =$

$t|_{U_x}$. Consider the set

$$S = \{(V, s) : V \subseteq U, s \in \mathcal{F}(V), \psi(V)(s) = t|_V\}.$$

Since $(U_x, s_{U_x}) \in S$, we see that S is nonempty. Define a partial order $\leqslant$ on S by $(U, s) \leqslant (V, t) \Leftrightarrow U \leqslant V$ and $t|_U = s$.

By the gluability of sheaves, we see that every chain in S has an upper bound. Thus, there exists a maximal element, say (V, s), in S, by Zorn's lemma. We aim to show that $V = U$. If not, there exists (W, r) such that $V \nsubseteq W \subseteq U$, $r \in \mathcal{F}(W)$ and $\psi(W)(r) = t|_W$. We may assume that $W \cap V \neq \varnothing$, otherwise, we are done. Note that

$$\psi(W \cap V)(s|_{W\cap V} - r|_{W\cap V}) = \psi(V)(s)|_{W\cap V} - \psi(W)(r)|_{W\cap V} = (t|_V)|_{W\cap V} - (t|_W)|_{W\cap V} = 0.$$

So, $s|_{W\cap V} - r|_{W\cap V} \in \ker \psi(W \cap V) = \operatorname{im} \varphi(W \cap V)$. This means that $s|_{W\cap V} - r|_{W\cap V} = \varphi(W \cap V)(u)$ for some $u \in \mathcal{F}'(W \cap V)$. Note that $\mathcal{F}'$ is flasque, the restricition map $\mathcal{F}'(W) \to \mathcal{F}'(W \cap V)$ is surjective. Thus, there exists $\tilde{u} \in \mathcal{F}'(W)$ such that $\tilde{u}|_{W\cap V} = u$. Consider the following commutative diagram

$$\begin{tikzcd}
\mathcal{F}'(W) \arrow[r, "\varphi(W)"] \arrow[d] & \mathcal{F}(W) \arrow[d] \\
\mathcal{F}'(W \cap V) \arrow[r, "\varphi(W\cap V)"] & \mathcal{F}(W \cap V)
\end{tikzcd}$$

We see that $\varphi(W)(\tilde{u})|_{W\cap V} = \varphi(W \cap V)(u) = s|_{W\cap V} - r|_{W\cap V}$. Thus, $s|_{W\cap V} = (\varphi(W)(\tilde{u}) + r)|_{W\cap V}$. Thus, by the gluability of sheaves, there exists a section $s' \in \mathcal{F}(W \cup V)$ such that $s'|_V = s$ and $s'|_W = \varphi(W)(\tilde{u}) + r$. Since $\psi(W \cup V)(s')|_V = t|_V$ and $\psi(W \cup V)(s')|_W = t|_W$, we see that $\psi(W \cup V)(s') = t|_{W\cup V}$. Thus, $(W \cup V, s') \in S$. This is a contradiction as $V \subsetneq W \cup V$. We conclude that (U, s) is the maximal element in S, i.e. $s \in \mathcal{F}(U)$ and $\psi(U)(s) = t|_U = t$. Thus, $\psi(U) : \mathcal{F}(U) \to \mathcal{F}''(U)$ is surjective. $\square$

Proposition 2.9. *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves and if $\mathcal{F}'$ and $\mathcal{F}$ are flasque, then $\mathcal{F}''$ is flasque.*

Proof. To show that $\mathcal{F}''$ is flasque, it suffices to prove that for any open subset $U \subseteq X$, the restricition map $\rho''_{XU} : \mathcal{F}''(X) \to \mathcal{F}''(U)$ is surjective. Since $\mathcal{F}'$ is flasque, we have a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}'(X) \arrow[r, "\varphi(X)"] \arrow[d, "\rho'_{XU}"] & \mathcal{F}(X) \arrow[r, "\psi(X)"] \arrow[d, "\rho_{XU}"] & \mathcal{F}''(X) \arrow[r] \arrow[d, "\rho''_{XU}"] & 0 \\
0 \arrow[r] & \mathcal{F}'(U) \arrow[r, "\varphi(U)"] & \mathcal{F}(U) \arrow[r, "\psi(U)"] & \mathcal{F}''(U) \arrow[r] & 0
\end{tikzcd}$$

Take any $a \in \mathcal{F}''(U)$, there exists $b \in \mathcal{F}(U)$ such that $\psi(U)(b) = a$. Since $\mathcal{F}$ is flasque, we see that $\rho_{XU} : \mathcal{F}(X) \to \mathcal{F}(U)$ is surjective. So, there exists some $c \in \mathcal{F}(X)$ such that $\rho_{XU}(c) = b$. Thus, we see that $\rho''_{XU}(\psi(X)(c)) = \psi(U)(\rho_{XU}(c)) = \psi(U)(b) = a$. Thus, $\rho''_{XU} : \mathcal{F}''(X) \to \mathcal{F}''(U)$ is surjective as desired. $\square$

Theorem 2.10. *If $\mathcal{F}$ is a flasque sheaf on a topological space X, then $\mathrm{H}^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

Proof. By Proposition 2.2, we may embed $\mathcal{F}$ in an injective sheaf of abelian groups $\mathcal{I}$. Let $\mathcal{G}$ be the quotient, then we have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0.$$

This short exact sequence induced a long exact sequence of cohomology, i.e.

$$\begin{aligned} 0 \to &\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{G}) \to \\ &\mathrm{H}^1(X, \mathcal{F}) \to \mathrm{H}^1(X, \mathcal{I}) \to \mathrm{H}^1(X, \mathcal{G}) \to \\ &\mathrm{H}^2(X, \mathcal{F}) \to \mathrm{H}^2(X, \mathcal{I}) \to \mathrm{H}^2(X, \mathcal{G}) \to \cdots \end{aligned}$$

Now since $\mathcal{F}$ is flasque, we have an exact sequence by Proposition 2.8,

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{G}) \to 0.$$

So we obtain a long exact sequence

$$0 \to \mathrm{H}^1(X, \mathcal{F}) \to \mathrm{H}^1(X, \mathcal{I}) \to \mathrm{H}^1(X, \mathcal{G}) \to \mathrm{H}^2(X, \mathcal{F}) \to \mathrm{H}^2(X, \mathcal{I}) \to \mathrm{H}^2(X, \mathcal{G}) \to \cdots$$

On the other hand, since $\mathcal{I}$ is injective, we have $\mathrm{H}^i(X, \mathcal{I}) = 0$ for $i > 0$. We see that $\mathrm{H}^1(X, \mathcal{F}) = 0$, and $\mathrm{H}^i(X, \mathcal{G}) = \mathrm{H}^{i+1}(X, \mathcal{F})$ for all $i \geqslant 1$. Note that $\mathcal{F}$ is flasque by hypothesis, $\mathcal{I}$ is flasque by Proposition 2.7, so $\mathcal{G}$ is flasque by Proposition 2.9. So by induction on i we get the result. ☐

2.2 A vanishing theorem of Grothendieck

Theorem 2.11 (Grothendieck). *Let X be a Noetherian topological space of dimension n. Then for all $i > n$ and all sheaves of abelian groups $\mathcal{F}$ on X, we have $\mathrm{H}^i(X, \mathcal{F}) = 0$.*

3 Čech cohomology

For a general space X, the sheaf cohomology groups may be quite difficult to compute – how does one produce a flasque or even injective resolution resolution in general? Fortunately, there is another construction of sheaf cohomology which, though cumbersome to define, is much more amenable to computation.

3.1 Motivation: the Mittag-Leffler problem

In this section, we motivate the definition of Čech cohomology with a classical problem originally studied by Mittag-Leffer. Let X be a Riemann surface, i.e. a one-dimensional complex manifold, which we may assume to be connected. Suppose E is a closed, discrete subset of X, i.e. E has no limit point in X. For each $a \in E$, we are given a function $z_a : U_a \to \mathbb{C}$ on some neighborhood $U_a \subseteq X$ of a such that $z_a(a) = 0$. Consider the function

$$p_a(z_a) = \sum_{j=1}^{m_a} \frac{\alpha_{aj}}{z_a^j}.$$

The **Mittag-Leffler problem** is to find a meromorphic function $f : X \to \mathbb{C}$ such that f is holomorphic on $X - E$ and for all $a \in E$, the function $f - p_a(z_a)$ has a removable singularity at a. Then $p_a(z_a)$ will be the principal part of f on U_a. Equivalently, we are asked to extend some meromorphic functions defined on open subsets in X to a meromorphic function on the whole Riemann surface. We can restate the problem as:

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X and suppose that $\{f_i : U_i \to \mathbb{C}\}$ is a collection of meromorphic functions defined on U_i such that either f_i is holomorphic on U_i or has a single point $a_i \in U_i$ with $a_i \notin U_j$ if $j \neq i$. The Mittag-Leffler problem is then to find a meromorphic function $f : X \to \mathbb{C}$ such that for each $i \in I$, $f|_{U_i} - f_i$ is holomorphic.

Let $\mathcal{M}$ be the sheaf of meromorphic functions on X. First notice that if f_i agree on all overlaps $U_i \cap U_j$, then the sheaf condition on $\mathcal{M}$ guarantees that there is a global meromorphic function $f \in \mathcal{M}(X)$ such that $f|_{U_i} = f_i$ for all i. In this case, we have $f|_{U_i} - f_i = 0$, a much stronger conclusion than Mittag-Leffler problem asks for. In general, if we can find a family of holomorphic functions $\{h_i : U_i \to \mathbb{C}\}$ on each U_i such that $(f_i + h_i))|_{U_i \cap U_j} = (f_j + h_j)|_{U_i \cap U_j}$ for all i, j, we can glue $f_i + h_i$ together to find the desired f. This can be rewritten as

$$f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j} = h_j|_{U_i \cap U_j} - h_i|_{U_i \cap U_j}.$$

Set $t_{ij} = f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j}$. Then we have $t_{ij} \in \mathcal{O}(U_i \cap U_j)$, where $\mathcal{O}$ is the sheaf of holomorphic functions on X, if the above equation is satisfied. Moreover, when restricting on $U_i \cap U_j \cap U_k$ for any i, j and k, we have

$$t_{jk} - t_{ik} + t_{ij} = 0.$$

Thus, we want to find holomorphic functions $h_i \in \mathcal{O}(U_i)$ such that

(1) $t_{ij} = h_j - h_i$ on $U_i \cap U_j$ for any i, j and

(2) $t_{jk} - t_{ik} + t_{ij} = 0$ on $U_i \cap U_j \cap U_k$ for any i, j and k.

Definition 3.1. *Let X be a Riemann surface, $\mathcal{U}$ an open cover of X and $\mathcal{O}$ be the sheaf of holomorphic functions. A family of sections $(t_{ij}) \in \prod_{ij} \mathcal{O}(U_i \cap U_j)$ is called a* **Čech 1-cocycle** *if for all i, j, k, we have $t_{jk} - t_{ik} + t_{ij} = 0$ on $U_i \cap U_j \cap U_k$. Under component-wise addtion, the set of 1-cocycle forms a group, denoted by $\check{Z}(\mathcal{U}, \mathcal{O})$.*

Definition 3.2. *A family of sections $(t_{ij}) \in \prod_{ij} \mathcal{O}(U_i \cap U_j)$ is called a* **Čech 1-coboundary** *if there exists a family $(h_i) \in \prod_i \mathcal{O}(U_i)$ such that $t_{ij} = h_j - h_i$ on each $U_i \cap U_j$. This forms a subgroup of $\check{Z}(\mathcal{U}, \mathcal{O})$, which is denoted by $\check{B}(\mathcal{U}, \mathcal{O})$.*

Definition 3.3. *The **first** Č**ech cohomology group of the cover** $\mathcal{U}$ with coefficients in $\mathcal{O}$ is the quotient group*

$$\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}) = \check{Z}(\mathcal{U}, \mathcal{O}) / \check{B}(\mathcal{U}, \mathcal{O}).$$

Now, to solve the Mittag-Leffler problem, it suffices to investigate that whether we have $\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}) = 0$ for a Riemann surface X with cover $\mathcal{U}$.

3.2 Čech cohomology of an open cover

In previous section, we defined the first Čech cohomology for a Riemann surface X with an open cover $\mathcal{U}$. We can generalize this to any space X with an open cover $\mathcal{U} = (U_j)_{j \in J}$.

In this section, we fix a topological space X and a presheaf $\mathcal{F}$ on X. Let $\mathcal{U} = (U_j)_{j \in J}$ be an open cover of X, where J is an index set. Before we step into our main result, we make some conventions first for convenience.

Notation.

- X: *a topological space.*
- $\mathcal{U} = (U_j)_{j\in J}$: *an open cover of* X*, where* J *is an index set.*
- R: *a fixed commutative unitary ring.*
- $\mathcal{F}$: *a presheaf of* R*-modules on* X*.*
- $I = (i_0, \cdots, i_p)$: *a* $(p+1)$*-tuple of elements of* J*, where* $p \geqslant 0$ *and* $i_k \in J$ *are not necessarily distinct.*

Definition 3.4. *Let* $I = (i_0, \cdots, i_p)$ *be a* $(p+1)$*-tuple of elements of* J*. We define an open subset* U_I *to be the intersection of open subsets in* $\mathcal{U}$ *with subscripts in* I*, i.e.*

$$U_I = U_{i_0,\cdots,i_p} = U_{i_0} \cap \cdots \cap U_{i_p}.$$

We define $U_{i_0,\cdots,\widehat{i_j},\cdots,i_p}$ *to be the intersection*

$$U_{i_0,\cdots,\widehat{i_j},\cdots,i_p} = U_{i_0} \cap \cdots \cap U_{i_{j-1}} \cap U_{i_{j+1}} \cap \cdots \cap U_{i_p}$$

of the p *subsets with* U_{i_j} *excluded.*

Remark. *By definition,* $U_{i_0,\cdots,i_p} \subseteq U_{i_0,\cdots,\widehat{i_j},\cdots,i_p}$ *induces an inclusion map*

$$\delta^p_j : U_{i_0,\cdots,i_p} \hookrightarrow U_{i_0,\cdots,\widehat{i_j},\cdots,i_p}$$

Example 3.5. *As the following picture shows, we see that* $U_{i_0 i_1 i_2 i_3} \subseteq U_{i_0 i_1 \widehat{i_2} i_3}$*.*

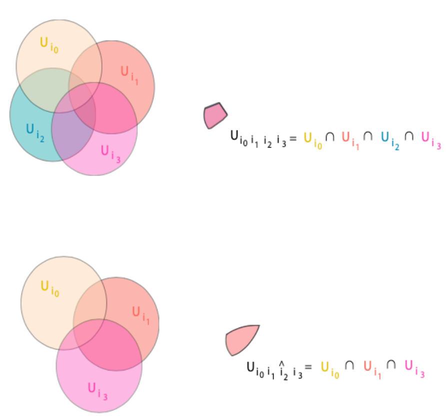

Figure 1: An illustration of $U_{i_0 i_1 i_2 i_3}$ and $U_{i_0 i_1 \widehat{i_2} i_3}$.

To introduce Čech cohomology, we first construct a cochain complex. The idea to construct the desired complex arises from the Mittag-Leffler problem. More precisely, let $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$. By the construction in the previous section, we have a sequence

$$0 \to \mathcal{F}(X) \xrightarrow{d_0} \prod_{j\in J} \mathcal{F}(U_j) \xrightarrow{d_1} \prod_{(i,j)\in J^2} \mathcal{F}(U_i \cap U_j) \xrightarrow{d_2} \prod_{(i,j,k)\in J^3} \mathcal{F}(U_i \cap U_j \cap U_k),$$

where $d_0 : s \mapsto (s|_{U_j})_j$, $d_1 : (s_j)_j \mapsto (s_i|_{U_{ij}} - s_j|_{U_{ij}})_{i,j}$ and $d_3 : (s_{ij})_{i,j} \mapsto (s_{jk}|_{U_{ijk}} - s_{ik}|_{U_{ijk}} + s_{ij}|_{U_{ijk}})$. So, we can extend this sequence to obtain a cochain comlex.

Definition 3.6. *Given a topological space X, an open cover $\mathcal{U} = (U_j)_{j\in J}$ of X, and a presheaf of abelian groups $\mathcal{F}$ on X, the R-module of* Čech p-cochains *$C^p(\mathcal{U}, \mathcal{F})$ is the set of all functions f with domain J^{p+1} such that $f(i_0, \cdots, i_p) \in \mathcal{F}(U_{i_0\cdots i_p})$; in other words,*

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{(i_0,\cdots,i_p)\in J^{p+1}} \mathcal{F}(U_{i_0,\cdots,i_p}),$$

the set of all J^{p+1}-indexed families $(f_{i_0,\cdots,i_p})_{(i_0,\cdots,i_p)} \in J^{p+1}$ with $f_{i_0,\cdots,i_p} \in \mathcal{F}(U_{i_0\cdots i_p})$.

Example 3.7. *If $p = 0$, we have*

$$C^0(\mathcal{U}, \mathcal{F}) = \prod_{j\in J} \mathcal{F}(U_j),$$

i.e. a 0-cochain is a J-indexed family $f = (f_j)_{j\in J}$ with each $f_j \in \mathcal{F}(U_j)$.

If $p = 1$, we have

$$C^1(\mathcal{U}, \mathcal{F}) = \prod_{(i,j)\in J^2} \mathcal{F}(U_i \cap U_j),$$

i.e. a 1-cochain is a J^2-indexed family $f = (f_{i,j})_{(i,j)\in J^2}$ with $f_{i,j} \in \mathcal{F}(U_i \cap U_j)$.

Remark. *Note that $\mathcal{F}(\varnothing) = 0$, we may assume that $U_{i_0,\cdots,i_p} \neq \varnothing$. Indeed, if $U_{i_0,\cdots,i_p} = \varnothing$, the component corresponding to the tuple $(i_0, \cdots, i_p)$ is trivial, which means that we could just omit the component with $U_{i_0,\cdots,i_p} = \varnothing$.*

Remark. *Recall that a presheaf is just a contravariant functor, we see that the restriction map*

$$\rho^{U_{i_0,\cdots,\widehat{i_j},\cdots,i_p}}_{U_{i_0,\cdots,i_p}} : \mathcal{F}(U_{i_0,\cdots,\widehat{i_j},\cdots,i_p}) \to \mathcal{F}(U_{i_0,\cdots,i_p})$$

is induced by the inclusion map $\delta^p_j : U_{i_0,\cdots,i_p} \hookrightarrow U_{i_0,\cdots,\widehat{i_j},\cdots,i_p}$.

For simplicity, we denote that restriction map $\rho^{U_{i_0,\cdots,\widehat{i_j},\cdots,i_p}}_{U_{i_0,\cdots,i_p}}$ by $\rho^j_{i_0,\cdots,i_p}$ or just $\mathcal{F}(\delta^p_j)$.

Now, to obtain a cochain complex, it remains to construct the coboundary maps.

Definition 3.8. *Given a topological space X, an open cover $\mathcal{U} = (U_j)_{j\in J}$ of X, and a presheaf of R-modules $\mathcal{F}$ on X, the* **coboundary maps** *$\delta^p_{\mathcal{F}} : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ are given by*

$$\delta^p_{\mathcal{F}} = \sum_{j=0}^{p+1} (-1)^j \mathcal{F}(\delta^{p+1}_j)$$

on each component $\mathcal{F}(U_{i_0,\cdots,\widehat{i_j},\cdots,i_{p+1}})$. Explicitly, for each p-cochain $f \in C^p(\mathcal{U}, \mathcal{F})$, and any sequence $I = (i_0, \cdots, i_{p+1}) \in J^{p+2}$, we define

$$(\delta^p_{\mathcal{F}} f)_{i_0,\cdots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \rho^j_{i_0,\cdots,i_{p+1}}(f_{i_0,\cdots,\widehat{i_j},\cdots,i_{p+1}}).$$

By a direct computation, we have that following proposition.

Proposition 3.9. *$\delta_{\mathcal{F}}^{p+1} \circ \delta_{\mathcal{F}}^{p} = 0$ for all $p \geqslant 0$.*

So, we obtain a cochain complex $(C^{\bullet}(\mathcal{U}, \mathcal{F}), \delta_{\mathcal{F}}^{\bullet})$. We now can define its cohomology.

Definition 3.10. *Given a topological space X, an open cover $\mathcal{U} = (U_j)_{j \in J}$ of X, and a presheaf $\mathcal{F}$ of R-modules on X, the R-module $B^p(\mathcal{U}, \mathcal{F})$ of* ***Čech p-boundaries*** *is given by*

$$B^p(\mathcal{U}, \mathcal{F}) = \operatorname{Im} \delta_{\mathcal{F}}^{p-1}$$

for $p \geqslant 1$ with $B^0(\mathcal{U}, \mathcal{F}) = 0$, and the R-module $Z^p(\mathcal{U}, \mathcal{F})$ of ***Čech p-cocycles*** *is given by*

$$Z^p(\mathcal{U}, \mathcal{F}) = \ker \delta_{\mathcal{F}}^{p}$$

, for $p \geqslant 0$.

Definition 3.11. *Given a topological space X, an open cover $\mathcal{U} = (U_j)_{j \in J}$ of X, and a presheaf $\mathcal{F}$ of R-modules on X, the* ***Čech cohomology groups*** *$\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F})$ of the cover $\mathcal{U}$ with values in $\mathcal{F}$ are defined by*

$$\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) = Z^p(\mathcal{U}, \mathcal{F}) / B^p(\mathcal{U}, \mathcal{F})$$

for each $p \geqslant 0$.

Theorem 3.12. *Given a topological space X, an open cover $\mathcal{U} = (U_j)_{j \in J}$ of X, and a presheaf of R-modules $\mathcal{F}$ on X, if $\mathcal{F}$ is a sheaf, then*

$$\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X) = \Gamma(X, \mathcal{F})$$

the global section of $\mathcal{F}$.

Proof. Recall that we have a left exact sequence

$$0 \to \mathcal{F}(X) \xrightarrow{d_0} \prod_{j \in J} \mathcal{F}(U_j) \xrightarrow{d_1} \prod_{(i,j) \in J^2} \mathcal{F}(U_i \cap U_j),$$

where $d_0 : s \mapsto (s|_{U_j})_j$ and $d_1 : (s_j)_j \mapsto (s_i|_{U_{ij}} - s_j|_{U_{ij}})_{i,j}$. By definition, we see that $\delta_{\mathcal{F}}^0 = d_1$ and $\ker \delta_{\mathcal{F}}^0 = \ker d_1 = \operatorname{Im} d_0 = \mathcal{F}(X)$. Thus, $\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X) = \Gamma(X, \mathcal{F})$ is the global section of $\mathcal{F}$. □

3.3 Čech cohomology with values in a presheaf

We now want to give a partial order between two open covers

Definition 3.13. *Given two open covers $\mathcal{U} = (U_i)_{i \in I}$ and $\mathcal{V} = (V_j)_{j \in J}$ of a space X, we say that $\mathcal{V}$ is a* ***refinement*** *of $\mathcal{U}$, denoted $\mathcal{U} \prec \mathcal{V}$, if there is a function $\tau : J \to I$ such that*

$$V_j \subseteq U_{\tau(j)} \qquad \text{for all } j \in J.$$

Two covers $\mathcal{U}$ and $\mathcal{V}$ are said to be ***equivalent*** *if $\mathcal{V} \prec \mathcal{U}$ and $\mathcal{U} \prec \mathcal{V}$.*

Example 3.14. *Let $\mathcal{U} = \{U_1, U_2, U_3\}$. Let $\mathcal{V} = \{V_1, V_2, V_3, V_4, V_5, V_6\}$. Then $\mathcal{U} \prec \mathcal{V}$ with $\tau : \{1,2,3,4,5,6\} \to \{1,2,3\}$ where $\tau(1) = 1$, $\tau(2) = 1$, $\tau(3) = 2$, $\tau(4) = 2$, $\tau(5) = 3$, $\tau(6) = 3$ since $V_1 \subseteq U_1, V_2 \subseteq U_1$, $V_3 \subseteq U_2$, $V_4 \subseteq U_2, V_5 \subseteq U_3, V_6 \subseteq U_3$.*

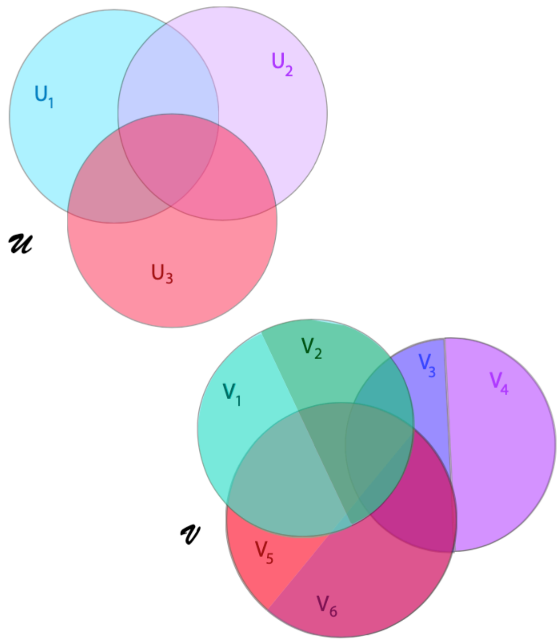

Figure 2: An illustration of $\mathcal{U} \prec \mathcal{V}$.

Definition 3.15. *Let $\tau : J \to I$ be a function such that*

$$V_j \subseteq U_{\tau(j)} \qquad \text{for all } j \in J.$$

We can define a homomorphism $\tau^p : C^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{V}, \mathcal{F})$ as follows: for every p-cochain $f \in C^p(\mathcal{U}, \mathcal{F})$, let $\tau^p f \in C^p(\mathcal{V}, \mathcal{F})$ be the p-cochain given by

$$(\tau^p f)_{j_0 \cdots j_p} = \rho_V^U(f_{\tau(j_0) \cdots \tau(j_p)})$$

for all $(j_0, \cdots, j_p) \in J^{p+1}$, where ρ_V^U denotes the restriction map associated with the inclusion of $V_{j_0 \cdots j_p}$ into $U_{\tau(j_0) \cdots \tau(j_p)}$.

By direct computation, we see that the map $\tau^p : C^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{V}, \mathcal{F})$ commutes with $\delta_{\mathcal{F}}$ so

$$\tau^* : C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{V}, \mathcal{F})$$

is a chain map. Thus, we have a homomorphism $\tau^{*p} : \check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^p(\mathcal{V}, \mathcal{F})$

Proposition 3.16. *Given any two open covers U and V of a space X, if $\mathcal{U} \prec \mathcal{V}$ and if $\tau_1 : J \to I$ and $\tau_2 : J \to I$ are functions such that*

$$V_j \subseteq U_{\tau_1(j)} \text{ and } V_j \subseteq U_{\tau_2(j)} \text{ for all } j \in J,$$

then $\tau_1^{*p} = \tau_2^{*p}$ for all $p \geqslant 0$.

Proof. The ideal is to construct a chain homotopy. Given any $f \in C^p(\mathcal{U}, \mathcal{F})$, let

$$(k^p f)_{j_0 \cdots j_{p-1}} = \sum_{h=0}^{p=1} (-1)^h \rho_h (f_{\tau_1(j_0) \cdots \tau_1(j_h) \tau_2(j_h) \cdots \tau_2(j_{p-1})})$$

for all $(j_0, \cdots, j_{p-1}) \in J^p$, where ρ_h denotes the restriction map associated with the inclusion of $V_{j_0 \cdots j_{p-1}}$ into $U_{\tau_1(j_0) \cdots \tau_1(j_h) \tau_2(j_h) \cdots \tau_2(j_{p-1})}$. Then, by a direct computation, we see that

$$d_{\mathcal{F}}^{p-1} \circ k^p(f) + k^{p+1} \circ \delta_{\mathcal{F}}^p(f) = \tau_2^p(f) - \tau_1^p(f),$$

where $d_{\mathcal{F}}^p : C^p(\mathcal{V}, \mathcal{F}) \to C^{p+1}(\mathcal{V}, \mathcal{F})$ and $\delta_{\mathcal{F}}^p : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$.

Thus, $\tau_1^{*p} = \tau_2^{*p}$ for all $p \geqslant 0$. □

This proposition gives us a homomorphism $\rho_{\mathcal{V}}^{\mathcal{U}} : \check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^p(\mathcal{V}, \mathcal{F})$. Moreover, this partial order is directed. Indeed, we have given any two covers $\mathcal{U} = (U_i)_{i \in I}$ and $\mathcal{V} = (V_j)_{j \in J}$, the cover $\mathcal{W} = (U_i \cap V_j)_{(i,j) \in I \times J}$ is a common refinement of both $\mathcal{U}$ and $\mathcal{V}$, so $\mathcal{U} \prec \mathcal{W}$ and $\mathcal{V} \prec \mathcal{W}$. Again, by this proposition, we see that if $\mathcal{U} \prec \mathcal{V} \prec \mathcal{W}$, then

$$\rho_{\mathcal{W}}^{\mathcal{U}} = \rho_{\mathcal{W}}^{\mathcal{V}} \circ \rho_{\mathcal{V}}^{\mathcal{U}}$$

and

$$\rho_{\mathcal{U}}^{\mathcal{U}} = \mathrm{id}\,.$$

Now, if $\mathcal{U}$ and $\mathcal{V}$ are equivalent, we see that $\rho_{\mathcal{U}}^{\mathcal{V}} \circ \rho_{\mathcal{V}}^{\mathcal{U}} = \mathrm{id}$ and $\rho_{\mathcal{V}}^{\mathcal{U}} \circ \rho_{\mathcal{U}}^{\mathcal{V}} = \mathrm{id}$, i.e.

$$\rho_{\mathcal{V}}^{\mathcal{U}} : \check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^p(\mathcal{V}, \mathcal{F})$$

is an isomorphism.

Consequently, it appears that the family $(\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}))_{\mathcal{U}}$ is a direct system of R-modules indexed by the directed set of open covers of X.

Definition 3.17. *Let X be a topological space and $\mathcal{F}$ be a presheaf of R-modules. The p-**th Čech cohomology group** of X with values in $\mathcal{F}$ is defined to be the direct limit*

$$\check{\mathrm{H}}^p(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}).$$

3.4 Some properties of Čech cohomology

Proposition 3.18. *For every space X and every open cover $\mathcal{U}$ of X, the functor $C^p(\mathcal{U}, -)$ from presheaves to abelian groups is exact for all $p \geqslant 0$.*

Proof. If

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is an exact sequence of presheaves, then every sequence

$$0 \longrightarrow \mathcal{F}'(U_{i_0 \cdots i_p}) \longrightarrow \mathcal{F}(U_{i_0 \cdots i_p}) \longrightarrow \mathcal{F}''(U_{i_0 \cdots i_p}) \longrightarrow 0$$

is exact. Since exactness is preserved under direct products, we see that the sequence

$$0 \longrightarrow \prod_{(i_0,\cdots,i_p)} \mathcal{F}'(U_{i_0\cdots i_p}) \longrightarrow \prod_{(i_0,\cdots,i_p)} \mathcal{F}(U_{i_0\cdots i_p}) \longrightarrow \prod_{(i_0,\cdots,i_p)} \mathcal{F}''(U_{i_0\cdots i_p}) \longrightarrow 0,$$

i.e. the sequence

$$0 \longrightarrow C^p(\mathcal{U},\mathcal{F}') \longrightarrow C^p(\mathcal{U},\mathcal{F}) \longrightarrow C^p(\mathcal{U},\mathcal{F}'') \longrightarrow 0$$

is exact. $\square$

Corollary 3.19. *If*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is an exact sequence of presheaves, we have a long exact sequence of cohomology:

$$\cdots \longrightarrow \check{\mathrm{H}}^p(\mathcal{U},\mathcal{F}) \longrightarrow \check{\mathrm{H}}^p(\mathcal{U},\mathcal{F}'') \xrightarrow{d} \check{\mathrm{H}}^{p+1}(\mathcal{U},\mathcal{F}') \longrightarrow \check{\mathrm{H}}^{p+1}(\mathcal{U},\mathcal{F}) \longrightarrow \cdots,$$

where the coboundary operator d is defined as usual.

Proof. We have an exact sequence of complexes

$$0 \longrightarrow C^\bullet(\mathcal{U},\mathcal{F}') \longrightarrow C^\bullet(\mathcal{U},\mathcal{F}) \longrightarrow C^\bullet(\mathcal{U},\mathcal{F}'') \longrightarrow 0$$

This gives us a long exact sequence

$$\cdots \longrightarrow \check{\mathrm{H}}^p(\mathcal{U},\mathcal{F}) \longrightarrow \check{\mathrm{H}}^p(\mathcal{U},\mathcal{F}'') \xrightarrow{d} \check{\mathrm{H}}^{p+1}(\mathcal{U},\mathcal{F}') \longrightarrow \check{\mathrm{H}}^{p+1}(\mathcal{U},\mathcal{F}) \longrightarrow \cdots$$

Taking direct limit over all open covers, we obtain a long exact sequence

$$\cdots \longrightarrow \check{\mathrm{H}}^p(X,\mathcal{F}) \longrightarrow \check{\mathrm{H}}^p(X,\mathcal{F}'') \xrightarrow{d} \check{\mathrm{H}}^{p+1}(X,\mathcal{F}') \longrightarrow \check{\mathrm{H}}^{p+1}(X,\mathcal{F}) \longrightarrow \cdots$$

as direct limit functor is exact. $\square$

This is a pretty good result for presheaves, but this may not be true for sheaves. For sheaves, clearly, we have

Proposition 3.20. *For every space X and every open cover $\mathcal{U}$ of X, the functor $C^p(\mathcal{U},-)$ from sheaves to abelian groups is left exact for all $p \geqslant 0$.*

Now, we obtain a sequence of complexes

$$0 \longrightarrow C^\bullet(\mathcal{U},\mathcal{F}') \xrightarrow{\alpha} C^\bullet(\mathcal{U},\mathcal{F}) \xrightarrow{\beta} C^\bullet(\mathcal{U},\mathcal{F}'').$$

Consider the homomorphisms $\beta^p : C^p(\mathcal{U},\mathcal{F}) \to C^p(\mathcal{U},\mathcal{F}'')$, which need not be surjective. Denote by $C_0^p(\mathcal{U},\mathcal{F}'')$ the image of this homomorphism. We now have a complex $C_0^\bullet(\mathcal{U},\mathcal{F}'')$, which is a subcomplex of $C^\bullet(\mathcal{U},\mathcal{F}'')$, whose p-th cohomology groups will be denoted by $\check{\mathrm{H}}_0^p(\mathcal{U},\mathcal{F}'')$. We have an exact sequence of complexes:

$$0 \longrightarrow C^\bullet(\mathcal{U},\mathcal{F}') \longrightarrow C^\bullet(\mathcal{U},\mathcal{F}) \longrightarrow C_0^\bullet(\mathcal{U},\mathcal{F}'') \longrightarrow 0$$

So, we have a long exact sequence of cohomology:

$$\cdots \longrightarrow \check{\mathrm{H}}^p(\mathcal{U},\mathcal{F}) \longrightarrow \check{\mathrm{H}}_0^p(\mathcal{U},\mathcal{F}'') \xrightarrow{d} \check{\mathrm{H}}^{p+1}(\mathcal{U},\mathcal{F}') \longrightarrow \check{\mathrm{H}}^{p+1}(\mathcal{U},\mathcal{F}) \longrightarrow \cdots,$$

where the coboundary operator d is defined as usual.

Now, we consider two open covers $\mathcal{U} = (U_i)_{i\in I}$ and $\mathcal{V} = (V_j)_{j\in J}$ such that there exists a function $\tau : J \to I$ with $V_j \subseteq U_{\tau(j)}$ for all $j \in J$, i.e. $\mathcal{U} \prec \mathcal{V}$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C^\bullet(\mathcal{U},\mathcal{F}') \arrow[r, "\alpha"] \arrow[d, "\tau^*"] & C^\bullet(\mathcal{U},\mathcal{F}) \arrow[r, "\beta"] \arrow[d, "\tau^*"] & C^\bullet(\mathcal{U},\mathcal{F}'') \arrow[d, "\tau^*"] \\
0 \arrow[r] & C^\bullet(\mathcal{V},\mathcal{F}') \arrow[r, "\alpha"] & C^\bullet(\mathcal{V},\mathcal{F}) \arrow[r, "\beta"] & C^\bullet(\mathcal{V},\mathcal{F}'')
\end{tikzcd}$$

, we see that that τ^* maps $C_0^\bullet(\mathcal{U},\mathcal{F}'')$ into $C_0^\bullet(\mathcal{V},\mathcal{F}'')$ and we obtain a homomorphism

$$\tau^{*p} : \check{\mathrm{H}}_0^p(\mathcal{U},\mathcal{F}'') \to \check{\mathrm{H}}_0^p(\mathcal{V},\mathcal{F}'')$$

for each $p \geqslant 0$. Using a similar argument as in Proposition 3.16, the homomorphisms τ^* are independent of the choice of the mapping τ.

Now, recall that direct limit of an exact sequence of direct system is exact if the index set is directed, i.e. the set of all open coverings is directed. We have a long exact sequence

$$\cdots \longrightarrow \check{\mathrm{H}}^p(X,\mathcal{F}) \longrightarrow \check{\mathrm{H}}_0^p(X,\mathcal{F}'') \xrightarrow{d} \check{\mathrm{H}}^{p+1}(X,\mathcal{F}') \longrightarrow \check{\mathrm{H}}^{p+1}(X,\mathcal{F}) \longrightarrow \cdots$$

We now need to know the relation between $\check{\mathrm{H}}_0(X,\mathcal{F}'')$ and $\check{\mathrm{H}}(X,\mathcal{F}'')$.

Lemma 3.21. *Let $\mathcal{U} = (U_i)_{i\in I}$ be a covering and let $f = (f_i)$ be an element of $C^0(\mathcal{U},\mathcal{F}'')$. There exists a covering $\mathcal{V} = (V_j)_{j\in J}$ and a mapping $\tau : J \to I$ such that $V_j \subseteq U_{\tau(j)}$ and $\tau f \in C_0^0(\mathcal{V},\mathcal{F}'')$.*

Proof. Let $J = X$. For any $x \in J = X$, take a $\tau x \in I$ such that $x \in U_{\tau x}$. Noticing that $f_{\tau x}$ is a section of $\mathcal{F}''$ over $U_{\tau x}$, there exists an open neighborhood V_x of x, contained in $U_{\tau x}$ and a section b_x of $\mathcal{F}$ over V_x such that $\beta(V_x)(b_x) = f_{\tau x}|_{V_x}$ on V_x. Indeed, $\beta : \mathcal{F} \to \mathcal{F}''$ is surjective. Then β_x is surjective for all $x \in X$. Let $s = f_{\tau x}$ and s_x be the image of s in $\mathcal{F}''_x$. Since β_x is surjective, there exists $t_x \in \mathcal{F}_x$ s.t. $\beta_x(t_x) = s_x$. By the property of direct limit, we know that there exists a neighborhood of x, say V'_x, and $t \in \mathscr{F}(V'_x)$ s.t. t_x is the image of t in $\mathcal{F}_x$, i.e. $\rho(t) = t_x$.

Consider the following commutative diagram

$$\begin{tikzcd}
\mathcal{F}(V'_x) \arrow[r, "\beta(V'_x)"] \arrow[d, "\rho"'] & \mathcal{F}''(V'_x) \arrow[d, "\rho'"] \\
\mathcal{F}_x \arrow[r, "\beta_x"] & \mathcal{F}''_x
\end{tikzcd}$$

we must have $\rho'(s|_{V'_x}) = s_x$. Also $\langle V'_x, \beta(V'_x)(t)\rangle$ and $\langle V'_x, s|_{V'_x}\rangle$ have the same image in $\mathcal{F}''_x$. So there exists a neighborhood V_x of x contained in V'_x such that $\beta(V_x)(t|_{V_x}) = \beta(V'_x)(t)|_{V_x} = (s|_{V'_x})|_{V_x} = s|_{V_x}$. Hence, let $b_x = t|_{V_x}$, we have that $\beta(V_x)(b_x) = s|_{V_x}$.

The $\{V_x\}_{x\in X}$ form a covering $\mathcal{V}$ of X, and the b_x form a 0-chain $b = (b_x)_x$ of $\mathcal{V}$ with values in $\mathcal{U}$; since $\tau f = \beta(b)$, we have that $\tau f \in C_0^0(\mathcal{V}, \mathcal{F}'')$. $\square$

Now, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C_0^0(\mathcal{U},\mathcal{F}'') \arrow[r, "d^0"] \arrow[d] & C_0^1(\mathcal{U},\mathcal{F}'') \arrow[r, "d^1"] \arrow[d] & C_0^2(\mathcal{U},\mathcal{F}'') \arrow[r, "d^2"] \arrow[d] & \cdots \\
0 \arrow[r] & C^0(\mathcal{U},\mathcal{F}'') \arrow[r, "\partial^0"] & C^1(\mathcal{U},\mathcal{F}'') \arrow[r, "\partial^1"] & C^2(\mathcal{U},\mathcal{F}'') \arrow[r, "\partial^2"] & \cdots
\end{tikzcd}$$

we have a morphism of complexes $C_0^\bullet(\mathcal{U},\mathcal{F}'') \to C^\bullet(\mathcal{U},\mathcal{F}'')$. This induces a homomorphism $\check{\mathrm{H}}_0^p(\mathcal{U},\mathcal{F}'') \to \check{\mathrm{H}}^p(\mathcal{U},\mathcal{F}'')$. By taking direct limit, we have a homomorphism $\check{\mathrm{H}}_0^p(X,\mathcal{F}'') \to \check{\mathrm{H}}^p(X,\mathcal{F}'')$.

Proposition 3.22. *The canonical homomorphism $\check{\mathrm{H}}_0^p(X,\mathcal{F}'') \to \check{\mathrm{H}}^p(X,\mathcal{F}'')$ is bijective for $p = 0$ and injective for $p = 1$.*

Proof. We first show that $\check{\mathrm{H}}_0^1(X,\mathcal{F}'') \to \check{\mathrm{H}}^1(X,\mathcal{F}'')$ is injective. An element of the kernel of this mapping may be represented by a 1-cocycle $z = (z_{j_0j_1}) \in C_0^1(\mathcal{U},\mathcal{F}'')$, i.e. $z \in \mathrm{Im}\ \partial^0$. Thus, there exists an $f = (f_j) \in C^0(\mathcal{U},\mathcal{F}'')$ with $\partial^0 f = z$; applying Lemma 3.21 to f yields a covering $\mathcal{V}$ such that $\tau f \in C_0^0(\mathcal{V},\mathcal{F}'')$. So, $d^0(\tau f) = \tau z$ via the map $d^0 : C_0^0(\mathcal{V},\mathcal{F}'') \to C_0^0(\mathcal{V},\mathcal{F}'')$. This means that for finer enough $\mathcal{U}$, we have $z \in \mathrm{Im}\ d^0$. Thus its image in $H_0^1(X,\mathcal{F}'')$ is 0.

Using a similar argument, we see that $\check{\mathrm{H}}_0^0(X,\mathcal{F}'') \to \check{\mathrm{H}}^0(X,\mathcal{F}'')$ is injective. Again, Lemma 3.21 just means that $\check{\mathrm{H}}_0^0(X,\mathcal{F}'') \to \check{\mathrm{H}}^0(X,\mathcal{F}'')$ is surjective by the definition of direct limit. $\square$

Corollary 3.23. *We have an exact sequence*

$$0 \longrightarrow \check{\mathrm{H}}^0(X,\mathcal{F}') \longrightarrow \check{\mathrm{H}}^0(X,\mathcal{F}) \longrightarrow \check{\mathrm{H}}^0(X,\mathcal{F}'') \xrightarrow{d} \check{\mathrm{H}}^1(X,\mathcal{F}') \longrightarrow \check{\mathrm{H}}^1(X,\mathcal{F}) \longrightarrow \check{\mathrm{H}}^1(X,\mathcal{F}'')$$

Corollary 3.24. *If*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is an exact sequence of sheaves and $\check{\mathrm{H}}^1(X,\mathcal{F}') = 0$, then $\Gamma(X,\mathcal{F}) \to \Gamma(X,\mathcal{F}'')$ is surjective.

Proof. If $\check{\mathrm{H}}^1(X,\mathcal{F}') = 0$, then we have

$$0 \longrightarrow \check{\mathrm{H}}^0(X,\mathcal{F}') \longrightarrow \check{\mathrm{H}}^0(X,\mathcal{F}) \longrightarrow \check{\mathrm{H}}^0(X,\mathcal{F}'') \longrightarrow 0$$

Now, by Theorem 3.12, if $\mathcal{F}$ is a sheaf, then $\check{\mathrm{H}}^0(\mathcal{U},\mathcal{F}) = \Gamma(X,\mathcal{F})$ the global section of $\mathcal{F}$. Thus, we obtain a short exact sequence

$$0 \longrightarrow \Gamma(X,\mathcal{F}') \longrightarrow \Gamma(X,\mathcal{F}) \longrightarrow \Gamma(X,\mathcal{F}'') \longrightarrow 0$$

$\square$

On paracompact spaces, we can extend Proposition 3.22 for all values of p. Recall that

Definition 3.25. *An open cover $\mathcal{U} = (U_i)_{i\in I}$ of a space X is **locally finite** if for any $x \in X$, there exists some neighbourhood V_x of x such that the set $\{i \in I : U_i \cap V_x \neq \varnothing\}$ is finite.*

Definition 3.26. *A topological space X is **paracompact**, i.e. X is Hausdorff and if any covering of X admits a refinement that is locally finite.*

Using a similar argument as in Lemma 3.21, we have the following result

Lemma 3.27. *Let $\mathcal{U} = (U_i)_{i \in I}$ be a covering and let $f = (f_{i_0 \cdots i_p})$ be an element of $C^p(\mathcal{U}, \mathcal{F}'')$. There exists a covering $\mathcal{V} = (V_j)_{j \in J}$ and a mapping $\tau : J \to I$ such that $V_j \subseteq U_{\tau(j)}$ and $\tau f \in C_0^p(\mathcal{V}, \mathcal{F}'')$.*

So, we have a analogous result

Proposition 3.28. *If X is paracompact, the canonical homomorphism*

$$\check{\mathrm{H}}_0^p(X, \mathcal{F}'') \to \check{\mathrm{H}}^p(X, \mathcal{F}'')$$

is bijective for all $p \geqslant 0$.

As a corollary, we have

Corollary 3.29. *If X is paracompact, we have a long exact sequence:*

$$\cdots \longrightarrow \check{\mathrm{H}}^p(X, \mathcal{F}) \longrightarrow \check{\mathrm{H}}^p(X, \mathcal{F}'') \xrightarrow{d} \check{\mathrm{H}}^{p+1}(X, \mathcal{F}') \longrightarrow \check{\mathrm{H}}^{p+1}(X, \mathcal{F}) \longrightarrow \cdots,$$

4 Comparison of Čech cohomology and sheaf cohomology

We first define a sheafified version of Čech complex. For any open subset $U_{i_0, \cdots, i_p} \subseteq X$, let $f^{i_0, \cdots, i_p} : U_{i_0, \cdots, i_p} \to X$ denote the inclusion map. Now given $X, \mathcal{U}, \mathcal{F}$ as previous, we construct a complex $\mathscr{C}^p(\mathcal{U}, \mathcal{F})$ of sheaves as follows. For each $p \geqslant 0$, let

$$\mathscr{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{(i_0, \cdots, i_p) \in J^{p+1}} f_*^{i_0, \cdots, i_p}(\mathcal{F}|_{U_{i_0, \cdots, i_p}}),$$

and define

$$d^p : \mathscr{C}^p \to \mathscr{C}^{p+1}$$

by the same formula as above.

For every open subset U of X, let $\mathcal{U}_{/U}$ denote the induced covering of U consisting of all open subsets of the form $U_i \cap U$ with $U_i \in \mathcal{U}$.

Proposition 4.1. *Let $\mathcal{F}$ be a sheaf of R-modules on X. For any open subset U of X, we have*

$$\mathscr{C}^p(\mathcal{U}, \mathcal{F})(U) = C^p(\mathcal{U}_{/U}, \mathcal{F}).$$

Proof. For any subseteq U of X, we have

$$\begin{aligned}
\mathscr{C}^p(\mathcal{U}, \mathcal{F})(U) &= \prod_{(i_0,\cdots,i_p)\in J^{p+1}} f_*^{i_0,\cdots,i_p}(\mathcal{F}|_{U_{i_0,\cdots,i_p}})(U) \\
&= \prod_{(i_0,\cdots,i_p)\in J^{p+1}} \mathcal{F}|_{U_{i_0,\cdots,i_p}}((f^{i_0,\cdots,i_p})^{-1}(U)) \\
&= \prod_{(i_0,\cdots,i_p)\in J^{p+1}} \mathcal{F}|_{U_{i_0,\cdots,i_p}}(U \cap U_{i_0,\cdots,i_p}) \\
&= \prod_{(i_0,\cdots,i_p)\in J^{p+1}} \mathcal{F}(U \cap U_{i_0,\cdots,i_p}) \\
&= C^p(\mathcal{U}_{/U}, \mathcal{F}).
\end{aligned}$$

□

Lemma 4.2. *If $\mathcal{U} = (U_i)_{i\in I}$ is an open cover of X and if $U_i = X$ for some index i, then for any presheaf $\mathcal{F}$ of R-modules, we have $\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.*

Proof. Take $\mathcal{V} = \{X\}$, then we see that $\mathcal{U} \prec \mathcal{V} \prec \mathcal{U}$, i.e. $\mathcal{U}$ is equivalent to $\mathcal{V}$. Thus, the map

$$\rho_{\mathcal{V}}^{\mathcal{U}} : \check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^p(\mathcal{V}, \mathcal{F})$$

is an isomorphism. The Čech complex $C^\bullet(\mathcal{V}, \mathcal{F})$ is

$$\begin{tikzcd}
0 \arrow[r] & C^0(\mathcal{V},\mathcal{F}) \arrow[r, "d^0"] \arrow[d, equal] & C^1(\mathcal{V},\mathcal{F}) \arrow[r, "d^1"] \arrow[d, equal] & C^2(\mathcal{V},\mathcal{F}) \arrow[r, "d^2"] \arrow[d, equal] & C^3(\mathcal{V},\mathcal{F}) \arrow[r] \arrow[d, equal] & \cdots \\
0 \arrow[r] & \mathcal{F}(X) \arrow[r, "0"] & \mathcal{F}(X) \arrow[r, "1"] & \mathcal{F}(X) \arrow[r, "0"] & \mathcal{F}(X) \arrow[r] & \cdots
\end{tikzcd}$$

So, we see that $\check{\mathrm{H}}^p(\mathcal{V}, \mathcal{F}) = 0$ for all $p > 0$. Thus, $\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$. □

Proposition 4.3. *For every open cover $\mathcal{U}$ of the space X, for every $\mathcal{F}$ of R-modules on X, the complex*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathscr{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^0} \mathscr{C}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d^1} \cdots \longrightarrow \mathscr{C}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{d^p} \mathscr{C}^{p+1}(\mathcal{U}, \mathcal{F}) \xrightarrow{d^{p+1}} \cdots$$

is a resolution of the sheaf $\mathcal{F}$.

Proof. It suffices to show that for every $x \in X$, the stalk sequence

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathscr{C}^0(\mathcal{U}, \mathcal{F})_x \xrightarrow{d^0} \mathscr{C}^1(\mathcal{U}, \mathcal{F})_x \xrightarrow{d^1} \cdots \longrightarrow \mathscr{C}^p(\mathcal{U}, \mathcal{F})_x \xrightarrow{d^p} \mathscr{C}^{p+1}(\mathcal{U}, \mathcal{F})_x \xrightarrow{d^{p+1}} \cdots$$

is exact. Since direct limits preserves exact sequences, it suffices to show that for every $x \in X$, there is a neighborhood V of x such that the sequence

$$0 \longrightarrow \mathcal{F}(W) \longrightarrow \mathscr{C}^0(\mathcal{U}_{/W}, \mathcal{F}) \xrightarrow{d^0} \mathscr{C}^1(\mathcal{U}_{/W}, \mathcal{F}) \xrightarrow{d^1} \cdots \longrightarrow \mathscr{C}^p(\mathcal{U}_{/W}, \mathcal{F}) \xrightarrow{d^p} \mathscr{C}^{p+1}(\mathcal{U}_{/W}, \mathcal{F}) \xrightarrow{d^{p+1}} \cdots$$

is exact for all open subsets $W \subseteq V$.

Pick $V = U_{i_0}$ for some open subset U_{i_0} containing x. Then for $W \subseteq V = U_{i_0}$, then open cover $\mathcal{U}_{/W} = \{U_i \cap W : U_i \in \mathcal{U}\}$ contains $W = W \cap U_{i_0}$. By the definition of sheaves,

$$0 \to \mathcal{F}(W) \xrightarrow{\varepsilon} \prod_{j \in J} \mathcal{F}(U_j \cap W) \xrightarrow{d^0} \prod_{(i,j) \in J^2} \mathcal{F}(U_i \cap U_j \cap W)$$

is exact. So, we see that the above sequence is exact at $\mathcal{F}(W)$ and $\mathscr{C}^0(\mathcal{U}_{/W})$. By Lemma 4.2, we see that $\check{\mathrm{H}}^p(\mathcal{U}_{/W}, \mathcal{F}) = 0$ for all $p > 0$. So, the above sequence is exact at $\mathscr{C}^p(\mathcal{U}_{/W})$ for all $p > 0$. □

Proposition 4.4. *For every space X, every open cover $\mathcal{U}$ of X, every sheaf $\mathcal{F}$ of R-modules on X and every $p \geqslant 0$, there is a homomorphism*

$$\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) \to \mathrm{H}^p(X, \mathcal{F})$$

from Čech cohommology to sheaf cohomology. Consequently, there is a homomorphism

$$\check{\mathrm{H}}^p(X, \mathcal{F}) \to \mathrm{H}^p(X, \mathcal{F})$$

for every $p \geqslant 0$.

Proof. We have a resolution $0 \to \mathcal{F} \to \mathscr{C}^\bullet(\mathcal{U}, \mathcal{F})$ of the sheaf and an injective resolution $0 \to \mathcal{F} \to I^\bullet$ of $\mathcal{F}$. By comparison theorem of the injective case, there exists a chain map $f^\bullet : \mathscr{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \mathbf{I}^\bullet$ lifting the identity $\mathrm{id} : \mathcal{F} \to \mathcal{F}$, i.e. we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, "\mathrm{id}"] & \mathscr{C}^0(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d, dotted, "f^0"] & \mathscr{C}^1(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d, dotted, "f^1"] & \cdots \arrow[r] & \mathscr{C}^p(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d, dotted, "f^p"] & \cdots \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathbf{I}^0 \arrow[r] & \mathbf{I}^1 \arrow[r] & \cdots \arrow[r] & \mathbf{I}^p \arrow[r] & \cdots
\end{tikzcd}$$

Moreover, $f^\bullet$ is unique up to homotopy. Applying the global section functor $\Gamma(X, -)$ gives a chain map

$$\Gamma(X, f^\bullet) : \Gamma(X, \mathscr{C}^\bullet(\mathcal{U}, \mathcal{F})) \to \Gamma(X, \mathbf{I}^\bullet)$$

of complexes of R-modules. This induces the homomorphisms on cohomology

$$\mathrm{H}^\bullet(\Gamma(X, f^\bullet)) : \mathrm{H}^\bullet(\Gamma(X, \mathscr{C}^\bullet(\mathcal{U}, \mathcal{F}))) \to \mathrm{H}^\bullet(\Gamma(X, \mathbf{I}^\bullet)) = \mathrm{H}^\bullet(X, \mathcal{F}).$$

Note that $\Gamma(X, \mathscr{C}^\bullet(\mathcal{U}, \mathcal{F})) = C^\bullet(\mathcal{U}, \mathcal{F})$, so $\mathrm{H}^\bullet(\Gamma(X, \mathscr{C}^\bullet(\mathcal{U}, \mathcal{F}))) = \mathrm{H}^\bullet(C^\bullet(\mathcal{U}, \mathcal{F})) = \check{\mathrm{H}}^\bullet(\mathcal{U}, \mathcal{F})$. So,

$$\mathrm{H}^\bullet(\Gamma(X, f^\bullet)) : \check{\mathrm{H}}^\bullet(\mathcal{U}, \mathcal{F}) \to \mathrm{H}^\bullet(X, \mathcal{F})$$

are the desired homomorphisms. □

Lemma 4.5. *If $f : X \to Y$ is a continuous map, and if $\mathcal{F}$ is a flasque sheaf on X, then $f_*\mathcal{F}$ is a flasque sheaf on Y.*

Proof. If suffices to show that for any open subset U of Y, the restriction map $(f_*\mathcal{F})(Y) \to (f_*\mathcal{F})(U)$ is surjective. But this map is $\mathcal{F}(X) \to \mathcal{F}(f^{-1}(U))$, which is clearly surjective as $\mathcal{F}$ is flasque. □

Lemma 4.6. *If $\{\mathcal{F}_i\}_{i\in I}$ is a family of flasque sheaves on X, then $\prod_{i\in I}\mathcal{F}_i$ is a flasque sheaf.*

Proof. For any open subset U of X, the restriction map $\prod_{i\in I}\mathcal{F}_i(X)\to\prod_{i\in I}\mathcal{F}_i(U)$ is the product of the map $\mathcal{F}_i(X)\to\mathcal{F}_i(U)$, which is surjective. Hence, $\prod_{i\in I}\mathcal{F}_i(X)\to\prod_{i\in I}\mathcal{F}_i(U)$ is surjective. □

Proposition 4.7. *Let X be a topological space and $\mathcal{F}$ is a sheaf of R-modules. For every open cover $\mathcal{U}$ of X, if the sheaf $\mathcal{F}$ is flasque, then*

$$\mathrm{H}^p(\mathcal{U},\mathcal{F})=0$$

for all $p>0$. Consequently, the functor $\check{\mathrm{H}}^p(\mathcal{U},-)$ are effaceble for all $p>0$.

Proof. We first prove that $\mathscr{C}^p(\mathcal{U},\mathcal{F})$ is a flasque sheaf for each $p\geqslant 0$. Indeed, by definition $\mathscr{C}^p(\mathcal{U},\mathcal{F})=\prod_{(i_0,\cdots,i_p)\in J^{p+1}} f_*^{i_0,\cdots,i_p}(\mathcal{F}|_{U_{i_0,\cdots,i_p}})$. Since $\mathcal{F}$ is flasque, we see that $\mathcal{F}|_{U_{i_0,\cdots,i_p}}$ is also flasque on $U_{i_0,\cdots,i_p}$. Recall that direct image preserves flasque sheaves, we see that each $f_*^{i_0,\cdots,i_p}(\mathcal{F}|_{U_{i_0,\cdots,i_p}})$ is flasque. Again, a product of flasque sheaves is flasque, so we see that $\mathscr{C}^p(\mathcal{U},\mathcal{F})$ is flasque.

Thus, $0\longrightarrow\mathcal{F}\longrightarrow\mathscr{C}^\bullet(\mathcal{U},\mathcal{F})$ is a resolution of $\mathcal{F}$ by flasque sheaves. We can use this resolution to compute $\mathrm{H}^\bullet(X,\mathcal{F})$. Thus, we see that

$$\mathrm{H}^p(X,\mathcal{F})=\check{\mathrm{H}}^p(\mathcal{U},\mathcal{F})$$

for all $p\geqslant 0$. But since $\mathcal{F}$ is flasque, we have $\mathrm{H}^p(X,\mathcal{F})=0$ for all $p>0$ by Theorem 2.10. It follows that $\check{\mathrm{H}}^p(\mathcal{U},\mathcal{F})=0$ for all $p>0$. □

5 Cohomology of schemes

5.1 Cohomology of Noetherian affine schemes

Proposition 5.1 (Krull's Theorem). *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal, M a finitely generated A-module and N a submodule of M. Then the $\mathfrak{a}$-adic topology on N is induced by the $\mathfrak{a}$-adic topology on N. In particular, for any $n>0$, there exists $k\geqslant n$ such that $\mathfrak{a}^nN\supseteq N\cap\mathfrak{a}^kM$.*

Definition 5.2. *Let A be a ring, $\mathfrak{a}\subseteq A$ an ideal and M an A-module. Then we define the following submodule of M*

$$\Gamma_{\mathfrak{a}}(M)=\{m\in M\mid \mathfrak{a}^nm=0\text{ for some }n>0\}.$$

In other words, $m\in\Gamma_{\mathfrak{a}}(M)$ if and only if its annihilator is an open ideal in the $\mathfrak{a}$-adic topology on A.

Let X be a topological space, $Z\subseteq X$ a closed subset and $\mathcal{F}$ a sheaf of abelian groups on X. Then recall that $\Gamma_Z(X,\mathcal{F})=\{s\in\mathcal{F}(X)|\mathrm{Supp}(s)\subseteq Z\}$ is a subgroup of $\mathcal{F}(X)$, and we have a subsheaf $\mathscr{H}^0_Z(\mathcal{F})$ of $\mathcal{F}$ defined by

$$\Gamma\left(V,\mathscr{H}^0_Z(\mathcal{F})\right)=\{s\in\mathcal{F}(V)\mid\mathrm{Supp}(s)\subseteq Z\cap V\}.$$

If $(X,\mathcal{O}_X)$ is a ringed space and $\mathcal{F}$ a sheaf of modules, then $\mathscr{H}^0_Z(\mathcal{F})$ is a submodule of $\mathcal{F}$.

Lemma 5.3. *Let A be a Noetherian ring, $\mathfrak{a} \subseteq A$ an ideal and M an A-module. Set $X = \operatorname{Spec} A$ and let $\mathcal{F} = \widetilde{M}$. Then there is a canonical isomorphism of sheaves of modules $\Gamma_{\mathfrak{a}}(M) = \mathscr{H}^0_Z(\mathcal{F})$ where $Z = V(\mathfrak{a})$.*

Definition 5.4. *We first verify that*

$$0 \to \mathscr{H}^0_Z$$

Lemma 5.5. *Let A be a Noetherian ring, $\mathfrak{a} \subseteq A$ an ideal of A, and let I be an injective A-module. Then the submodule $J = \Gamma_{\mathfrak{a}}(I)$ is also an injective A-module.*

Lemma 5.6. *Let I be an injective module over a noetherian ring A. Then for any $f \in A$ the canonical morphism $I \to I_f$ is surjective.*

Proposition 5.7. *Let A be a Noetherian ring and set $X = \operatorname{Spec} A$. If I is an injective A-module then the sheaf of modules $\widetilde{I}$ on X is flasque.*

Corollary 5.8. *Let X be a Noetherian scheme, $\mathcal{F}$ a quasi-coherent sheaf of modules on X. Then there is a monomorphism $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a flasque quasi-coherent sheaf of modules.*

Theorem 5.9 (Serre). *Let $\mathcal{F}$ be a quasi-coherent sheaf on an affine scheme X. Then for any $i > 0$ we have $\mathrm{H}^i(X, \mathcal{F}) = 0$.*

Corollary 5.10. *Let X be a scheme and $U \subseteq X$ an affine open subset. Then the additive functor $\Gamma(U, -) : \mathfrak{Qco}(X) \to \boldsymbol{Ab}$ is exact.*

Theorem 5.11 (Serre). *Let X be a Noetherian scheme. Then the following conditions are equivalent:*
(i) X is affine;
(ii) $\mathrm{H}^i(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves of modules $\mathcal{F}$ and $i > 0$;
(iii) $\mathrm{H}^1(X, \mathcal{I}) = 0$ for all coherent sheaves of ideals $\mathcal{I}$.

Theorem 5.12. *Let X be a Noetherian separated scheme, let $\mathcal{U}$ be an open affine cover of X, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for all $p \geqslant 0$, the natural maps*

$$\mathrm{H}^p(\Gamma(X, f^\bullet)) : \check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) \to \mathrm{H}^p(X, \mathcal{F})$$

are an isomorphisms.

Proof. For $p = 0$, this is Theorem 3.12.

Now, we consider the case $p > 0$. By Corollary 5.8, we can embed $\mathcal{F}$ in a flasque, quasi-coherent sheaf $\mathcal{G}$. Let $\mathcal{H}$ be the quotient, i.e. we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0.$$

Recall that an intersection of affine open subsets of a saparated scheme is affine, we see that $U_{i_0,\cdots,i_p}$ is affine for any $(i_0, \cdots, i_p) \in J^{p+1}$. Recall that the above short exact sequence induced a short exact sequence on global sections as $\mathcal{F}|_{U_{i_0,\cdots,i_p}}$ is quasi-coherent, i.e.

$$0 \to \mathcal{F}(U_{i_0,\cdots,i_p}) \to \mathcal{G}(U_{i_0,\cdots,i_p}) \to \mathcal{H}(U_{i_0,\cdots,i_p}) \to 0.$$

So, we see that the corresponding sequence of Čech complexes

$$0 \to C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{G}) \to C^\bullet(\mathcal{U}, \mathcal{H}) \to 0$$

is exact as taking products preserves exactness. Thus, we have a long exact sequence

$$\begin{aligned} 0 \to &\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H}) \to \\ &\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^1(\mathcal{U}, \mathcal{G}) \to \check{\mathrm{H}}^1(\mathcal{U}, \mathcal{H}) \to \\ &\check{\mathrm{H}}^2(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^2(\mathcal{U}, \mathcal{G}) \to \check{\mathrm{H}}^2(\mathcal{U}, \mathcal{G}) \to \cdots \end{aligned}$$

Since $\mathcal{G}$ is flasque, by Proposition 4.7, we see that $\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{G}) = 0$ for all $p > 0$. So, we have an exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G}) \longrightarrow \Gamma(X, \mathcal{H}) \longrightarrow \check{\mathrm{H}}^1(\mathcal{U}, \mathcal{F}) \longrightarrow 0$$

and isomorphisms

$$\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{H}) \xrightarrow{\sim} \check{\mathrm{H}}^{p+1}(\mathcal{U}, \mathcal{F})$$

for all $p \geqslant 1$.

Agian, the above exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

induced a long exact sequence of cohomology, i.e.

$$\begin{aligned} 0 \to &\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H}) \to \\ &\mathrm{H}^1(X, \mathcal{F}) \to \mathrm{H}^1(X, \mathcal{G}) \to \mathrm{H}^1(X, \mathcal{H}) \to \\ &\mathrm{H}^2(X, \mathcal{F}) \to \mathrm{H}^2(X, \mathcal{G}) \to \mathrm{H}^2(X, \mathcal{H}) \to \cdots \end{aligned}$$

Since $\mathcal{G}$ is flasque, by Theorem 2.10, we see that $\mathrm{H}^p(\mathcal{U}, \mathcal{G}) = 0$ for all $p > 0$. So, we have an exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G}) \longrightarrow \Gamma(X, \mathcal{H}) \longrightarrow \mathrm{H}^1(\mathcal{U}, \mathcal{F}) \longrightarrow 0$$

and isomorphisms

$$\mathrm{H}^p(\mathcal{U}, \mathcal{H}) \xrightarrow{\sim} \mathrm{H}^{p+1}(\mathcal{U}, \mathcal{F})$$

for all $p \geqslant 1$. Now, apply Five Lemma to the following commutative diagram,

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(X, \mathcal{F}) \arrow[r] & \Gamma(X, \mathcal{G}) \arrow[r] \arrow[d, equal] & \Gamma(X, \mathcal{H}) \arrow[r] \arrow[d, equal] & \check{\mathrm{H}}^1(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d, dashed, equal] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\ 0 \arrow[r] & \Gamma(X, \mathcal{F}) \arrow[r] & \Gamma(X, \mathcal{G}) \arrow[r] & \Gamma(X, \mathcal{H}) \arrow[r] & \mathrm{H}^1(\mathcal{U}, \mathcal{F}) \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

we conclude that $\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{F}) \cong \mathrm{H}^1(\mathcal{U}, \mathcal{F})$. Recall that $\mathcal{H} = \mathrm{Coker}(\mathcal{F} \to \mathcal{G})$ is also quasi-coherent. Now, by induction on $p \geqslant 1$, we see that $\check{\mathrm{H}}^{p+1}(\mathcal{U}, \mathcal{F}) \cong \check{\mathrm{H}}^p(\mathcal{U}, \mathcal{H}) \cong \mathrm{H}^p(\mathcal{U}, \mathcal{H}) \cong \mathrm{H}^{p+1}(\mathcal{U}, \mathcal{F})$. Thus, □

5.2 Cohomology of projective space

Lemma 5.13. *Let A be a ring and $n \geqslant 1$. Then $A[x_1, \cdots, x_n]_{x_1 \cdots x_n}$ is a free A-module on the basis $\{x_{i_1} \cdots x_{i_n} | i_1, \cdots, i_n \in \mathbb{Z}\}$.*